Distributional Growth Theory I:
Wealth Distribution and Neoclassical Models of Growth

R.N.Vaughan
Department of Economics
University College London
Gower Street
London WC1E 6BT

e-mail: r.vaughan@ucl.ac.uk

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1. Introduction

The paper attempts an integration of size distribution theory and the theory of economic growth; in the present case we restrict the analysis to neoclassical models of growth, with exogeneity of savings behaviour. The paper considers the interdependence of the evolution of the size distribution of wealth with the factor distribution of income. Models of wealth distribution have the most part (see literature review) assumed exogeneity of the factor prices which determine the behaviour of the individual agents (whether such be families or individual persons) which constitute the distribution. Evolution of the distribution to its equilibrium (if it exists) is therefore conditioned by ¯xity of what might be termed the macroparameters in this connection. On the hand, models of economic growth, whether they be the earlier neoclassical models of growth, or the more recent developments of endogenous growth theory, focus attention on the behaviour of aggregate quantities with little if any discussion of the microfoundations underlying such aggregates. It is true of course that representative individuals are constructed which appear to give greater credence to such modelling; but such "creatures" in general are modelled in relation to aggregate quantities, or are presumed to re°ect cross sectional attributes of the population.

In essence the problem of integrating distributional and growth model lies in recognizing the appropriate aggregator relations that are required for the construct of macrorelationships. It may turn out, of course, that for certain types of microbehaviour, macromodels which focus on what might be termed ¯rst moment relations, are reasonably accurate, but without investigation we cannot be certain whether they correctly model the relevant aspects of the economy. Likewise models of distributional behaviour, which incorporate stable macroparameters which may be in°uenced by the evolution of the distribution, may be criticised on grounds of consistency, if the evolution of the distribution is inconsistent with long run macro stability.

(to be completed, Literature Review)

2. The Basic Model

The basic model we shall consider is one in which individual agents die and may leave property to more than one successor agent. We leave open the question of whether the individual agent is to be viewed as a single person or group of individuals, the family. The model at this stage is sufficiently general to allow a number of interpretations as regards the agent. For pedagogic purposes the reader may assume individual agents are individual persons, and successors to this person are 'children'.

This model will allow us to consider a number of applications, in particular the effects of inheritance taxation, as well as wealth and income taxes on the distribution. For simplicity we initially assume that there is no dispersion with respect to the representative individual agent's saving; this assumption will be dropped in a subsequent section. The model may therefore be specified as follows:

(a) The Representative Individual Agent

The income of the agent is assumed to consist of a wage component, the same for all agents, and an interest component-linearly dependent on the amount of wealth owned by the family. We assume a proportional savings function, savings being added to the individual's wealth stock, of which there is no depreciation. Thus we have,

$$\frac{dk}{dt} = s(p + rk) = g(k) \qquad (1)$$

where $p =$ the wage income, $r =$ the rate of interest, $k =$ the individual's wealth, and s the savings propensity, $0 < s < 1$.

(b) The Demographic Assumptions and the Distribution Equation

We assume that families steadily accumulate property according to equation (1); in addition we introduce a simple demographic assumption that within a given time period a proportion of families who have a given amount of wealth, cease to exist; each family's wealth is then shared equally between n successor families, which previously did not exist. This assumption is subsequently modified below.

The equation governing the frequency distribution of wealth is then shown in the Appendix to be,

$$\frac{\partial h(k;t)}{\partial t} = -g(k)\frac{\partial h(k;t)}{\partial k} - (g_k(k) + \lambda)h(k;t) + \lambda n^2 h(nk;t) \qquad (2)$$

where,

$h(k;t)$ denotes the number of families owning wealth k at time t

λ denotes the proportion of families which become extinct in each period

n denotes the number of successor families.

(c) The Macroeconomic Dimension

The model remains incomplete until we specify the time paths of wages and the interest rate. The simplest assumption is to specify $p(t)$ and $r(t)$ directly as functions of time; which would include the standard constancy case. This can be an admissible procedure, if, e.g. in the constant case we were interested in the distribution which corresponded to a macroeconomic steady state. However, a steady state may not exist, or we may need to prove existence, on which case we need to consider the dependency of the macroparameters on the evolution of the distribution function.

Let $Y(t)$ denote the total income of the community, β is defined as the proportion of income which goes in interest payments, and the remainder $(1-\beta)Y(t)$ of income is given to families in the form of wage payments. β is assumed to be constant over time. Let $K(t)$ be the total capital of the community, and $L(t)$ the total Labour supply. We assume that each family provides one unit of Labour supply, independent of the wage and family wealth.

The rate of interest is given by,

$$r(t) = \beta \frac{Y(t)}{K(t)} \tag{3}$$

and the wage rate by,

$$p(t) = (1-\beta)\frac{Y(t)}{L(t)} \tag{4}$$

$K(t)$ and $L(t)$ are determined by the relations,

$$L(t) = \int_{k=0}^{k=1} h(k;t)dk \tag{5}$$

$$K(t) = \int_{k=0}^{k=1} kh(k;t)dk \tag{6}$$

i.e. labour is simply the sum of the unit supplies with respect to the number of families in the distribution; whilst aggregate capital is the sum of families wealth.

This still leaves aggregate income undetermined. We shall assume that a relationship exists between aggregate income, and aggregate capital and labour,

$$Y(t) = F(K(t); L(t)) \tag{7}$$

which we particularize to the Cobb-Douglas form for some of our analytic results.

A word may be expressed, on the simplifications adopted. As we shall see the assumptions have been standardised to form a connection with standard neoclassical growth theory. Important modifications may relate to at least two dimensions. Firstly, the responsiveness of Labour supply to the wage rate, and the way this manifests itself according to the different characteristics of families. Thus the average wage component may systematically deviate across wealth categories, both according to the wage paid, and the heterogeneity of families with respect to their labour supply and other characteristics. The important aspect here is whether the wage component to families should systematically vary with wealth, and the functional form this should take. As we see below, alternative functional forms for the wage component may be adopted, although analytic solution in such cases may be problematic.

3. Solution of the Basic Model

We may collect the above equations to give the following basic model,

$$\frac{\partial h(k;t)}{\partial t} = -g(k)\frac{\partial h(k;t)}{\partial k} - (g_k(k) + \lambda)h(k;t) + \lambda n^2 h(nk;t) \tag{8}$$

$$\frac{dk}{dt} = s(p + rk) = g(k) \tag{9}$$

$$p(t) = (1 - \beta)\frac{Y(t)}{L(t)};\ r(t) = \beta\frac{Y(t)}{K(t)} \tag{10}$$

$$K(t) = \int_{k=0}^{k=\infty} k h(k;t)dk;\ L(t) = \int_{k=0}^{k=\infty} h(k;t)dk \tag{11}$$

$$Y(t) = F(K(t); L(t)) \tag{12}$$

The difficulty in solving such a system even for this simple case is that in (8) to (12we have a mixed difference-differential equation, the coefficients of which, through the influence of r and p are themselves functions of the distribution $h(k;t)$, thus making the equation non-linear. The approach to the solution of this system we adopt relies on the choice of a suitable integral transform procedure.

We take the moment or Mellin transform of $h(k;t)$ defined by,

$$M(h(k;t)) = M(m;t) = \int_{k=0}^{k=\infty} k^m h(k;t)dk \tag{13}$$

We may note that $M(0;t) = L(t)$, and $M(1;t) = K(t)$. Transforming the system with the Mellin transform, we have,

$$\frac{dM(s;t)}{dt} = spmM(m - 1;t) + (srm - \lambda + \lambda n^{1-m})M(m;t) \tag{14}$$

$$p(t) = (1 - \beta)\frac{Y(t)}{M(0;t)};\ r(t) = \beta\frac{Y(t)}{M(1;t)} \tag{15}$$

$$Y(t) = F(M(1;t); M(0;t)) = M(1;t)^{\gamma} M(0;t)^{1-\gamma} \tag{16}$$

First we may solve for the time paths of $L(t)$ and $K(t)$; with these determined the paths of $Y(t)$, and then $p(t)$ and $r(t)$ can be determined from equations (15)-(16).

Letting $m = 0$ in equation (14) we have,

$$\frac{dM(0;t)}{dt} = \lambda(n-1)M(0;t) \tag{17}$$

Thus,

$$M(0;t) = M(0;0)e^{\lambda(n-1)t} \tag{18}$$

i.e. the number of families grows at the exponential rate $\lambda(n-1)$ determined solely by the number of families leaving the distribution in each period, together with the number of net family replacements.

Letting $m = 1$ in (14) we have,

$$\frac{dM(1;t)}{dt} = srM(1;t) + spM(0;t) \tag{19}$$

Now from (15)-(16),

$$r(t) = \beta\left[\frac{M(1;t)}{M(0;t)}\right]^{\gamma-1} \tag{20}$$

and,

$$p(t) = (1-\beta)\left[\frac{M(1;t)}{M(0;t)}\right]^{\gamma} \tag{21}$$

Thus subst. these values for r and p in (19) and then (18) in (19) we have,

$$\frac{dM(1;t)}{dt} = sM(1;t)^{\gamma}M(0;0)^{1-\gamma}e^{\lambda(n-1)(1-\gamma)t} \tag{22}$$

The solution of (22) is,

$$M(1;t)^{1-\gamma} = \frac{s}{\lambda(n-1)}M(0;0)^{1-\gamma}[e^{\lambda(n-1)(1-\gamma)t} - 1] + M(1;0)^{1-\gamma} \tag{23}$$

Thus collecting the results we have the solution for the evolution of the macroparameters as,

$$L(t) = L(0)e^{\lambda(n-1)t} \tag{24}$$

$$K(t)^{1-\gamma} = \frac{s}{\lambda(n-1)}L(0)^{1-\gamma}(e^{\lambda(n-1)(1-\gamma)t} - 1) + K(0)^{1-\gamma} \tag{25}$$

$$\left[\frac{K(t)}{L(t)}\right]^{1-\gamma} = \frac{s}{\lambda(n-1)}L(0)^{1-\gamma}(1 - e^{-\lambda(n-1)(1-\gamma)t}) + \left[\frac{K(0)}{L(0)}\right]^{1-\gamma}e^{-\lambda(n-1)(1-\gamma)t} \tag{26}$$

$$r(t) = \beta\left[\frac{s}{\lambda(n-1)}(1 - e^{-\lambda(n-1)(1-\gamma)t}) + \left[\frac{K(0)}{L(0)}\right]^{1-\gamma}e^{-\lambda(n-1)(1-\gamma)t}\right]^{-1} \tag{27}$$

$$p(t) = (1-\beta)\left[\frac{s}{\lambda(n-1)}(1 - e^{-\lambda(n-1)(1-\gamma)t}) + \left[\frac{K(0)}{L(0)}\right]^{1-\gamma}e^{-\lambda(n-1)(1-\gamma)t}\right]^{\frac{\gamma}{1-\gamma}} \tag{28}$$

These results should of course appear familiar, indeed the equation for the evolution of the first moment may be put in a more recognizable form if we work with the normalized moment,

$$m(1;t) = \frac{M(1;t)}{M(0;t)} \tag{29}$$

Since,

$$\frac{dm(1;t)}{dt} = \frac{1}{M(0;t)}\frac{dM(1;t)}{dt} - \frac{M(1;t)}{M(0;t)^2}\frac{dM(0;t)}{dt} \tag{30}$$

then subst. (17) and (19) in (30) we have,

$$\frac{dm(1;t)}{dt} = sm(1;t)^{\gamma} - \lambda(n-1)m(1;t) \tag{31}$$

i.e. the Solow(1956) growth equation where $m(1;t) = k(t) = K(t)/L(t)$.

We may note that to derive equations referring to the evolution of the first two moments of the density function, $h(k;t)$, i.e. $L(t)$ and $K(t)$, there is strictly no need to use the distribution equation (8) or (14) at all. We might have inferred from our basic demographic assumption that the time path of population would be $L(t) = L(0)e^{\lambda(n-1)t}$, which always holds however complex the other elements of the basic model become. The time path of $K(t)$ can be derived directly from equation (9) by multiplying the R.H.S. by $h(k;t)$ and integrating over k, and equating the L.H.S. with $dK(t)/dt$. This equation alone suffices to determine the time path of $K(t)$ since the linear system is moment separable, i.e. the evolution of the first moment $K(t)$ does not depend on higher moments of the distribution as a result of the linear savings function. Naturally we may only dispense with (8) or (14) for the first two moments; to determine the evolution of the higher moments the distribution equation is a necessity.

The equations for the higher moments may be derived from (14), letting n take on integer values, and after substituting in the values of $r(t)$ and $p(t)$ from (27), (28). Two approaches to the solution of these equations for the higher moments are evident. From equations (26)-(29) (and from our prior knowledge of the Solow equation) we note that although the first two moments of the distribution function do not approach limiting values, their ratio does, as do the values of $r(t)$ and $p(t)$. These limiting values are obtained irrespective of the values taken by the other moments of the distribution, and also irrespective of the initial distribution, at $t = 0$, of individuals over wealth ranges. The simplest approach is thus to assume that sufficient time has elapsed to ensure that there is macroeconomic equilibrium, that is $m(1;t)$, $r(t)$ and $p(t)$ are constant at their equilibrium values, i.e.,

$$m(1;t) = \left[\frac{s}{\lambda(n-1)}\right]^{\frac{1}{1-\gamma}} \tag{32}$$

$$r(t) = \frac{\beta\lambda(n-1)}{s} = r^{*} \tag{33}$$

$$p(t) = (1 - \beta)\left[\frac{s}{\lambda(n-1)}\right]^{\frac{\gamma}{1-\gamma}} = p^{*} \tag{34}$$

and thence subst. these relations directly into equation (14) and solve for the higher moments. The alternative is to substitute the time dependent relations (27) and (28) into (14) and then solve. This approach however leads to much additional work without commensurate gain or clarity as regards the solution to the problem. We therefore consider the moment solution where $r(t)$ and $p(t)$ are constant at their equilibrium values. Note that this solution is different from that adopted by assuming r and p were exogenously given. We have here shown that p and r tend to equilibrium within the context of the distributional model.

For constant p and r we may solve equation by recursion (see Appendix). The normalized moments are given by,

$$m(N;t) = \frac{M(N;t)}{M(0;t)} = \sum_{j=1}^{j=N} C_j \exp(c_j t) + \frac{N!(sp)^N}{\prod_{j=1}^{j=N}(d_j)} \tag{35}$$

where we define,

$$c_j = srj - \lambda + \lambda n^{1-j} - \lambda(n-1) \tag{36}$$

and,

$$d_j = \lambda(n-1) - (sjr - \lambda + \lambda n^{1-j}) \tag{37}$$

Whether stable values of the $m(N;t)$ exist or not depends on the sign of the term,

$$srj - \lambda + \lambda n^{1-j} - \lambda(n-1) \tag{38}$$

For low values of j this may be negative, and thus we have stability, but ultimately (38) will definitely become positive as j takes on larger and larger values. We thus have a solution for a limiting distribution in which the lower moments of the distribution are stable whilst the higher moments are not.

Major interest centres around the change in the second normalized moment and the variance. We have,

$$m(N;t) = \frac{M(N;t)}{M(0;t)} = C_1 \exp(sr - \lambda(n-1))t + C_2 \exp(2sr - \lambda + \lambda n^{-1} - \lambda(n-1))t + \frac{2(sp)^2}{(\lambda(n-1) - sr)(\lambda(n-1) - 2sr - \lambda + \lambda n^{-1})} \tag{39}$$

Since we are in macro equilibrium, from (33),

$$sr = \beta\lambda(n-1) \tag{40}$$

where of course, $0 < \beta < 1$.

Thus the exponential associated with C_1 tends to zero for large t; for a stable variance we require that the exponential associated with C_2 tends to zero.

Thus if,

$$2sr - \lambda n + \lambda n^{-1} = \lambda(n-1)(2\beta - (1/n) - 1) > 0 \tag{41}$$

then,

$$m(2;t) \rightarrow C_2 \exp(\lambda(n-1)(2\beta - (1/n) - 1)t) \tag{42}$$

If,

$$\lambda(n-1)(2\beta - (1/n) - 1) < 0 \tag{43}$$

then,

$$m(2;t) \rightarrow \frac{2(sp)^2}{(\lambda(n-1) - sr)(\lambda(n-1)(1 + (1/n) - 2\beta)} \tag{44}$$

Since we remember that β is the share of national income that is paid out in interest payments, and remembering that $\beta < 0.5$ in most advanced capitalist countries, investigation of the stable branch of $m(2;t)$ is not wholly without relevance. The term for the second moment may be considerably simplified by substituting in the equilibrium values p^* and r^*; we have,

$$m(2;t) \rightarrow \frac{2(1-\beta)}{(1 + \frac{1}{n} - 2\beta)}(k^*)^2 \tag{45}$$

where k^* is the equilibrium value of mean wealth of the population defined by (32).

The variance of the equilibrium distribution of wealth is given by,

$$var(k) = m(2;t) - m(1;t)^2 = \frac{(1 - \frac{1}{n})}{(1 + \frac{1}{n} - 2\beta)}(k^*)^2 \tag{46}$$

The square of the coefficient of variation has quite an interesting interpretation, for large n, then we have,

$$(c.var)^2 \approx \frac{1}{(1-\beta) - \beta} \tag{47}$$

i.e. it is equal to the reciprocal of the difference between the proportion of national income going to wages and the proportion going to profit.

With regard to comparative statics, we may consider the effects of changing the share of profit, and the number of children on the variance and coefficient of variation. We may note that whilst β has no influence on the mean wealth, due to the proportional nature of the savings function, it does have an effect on the variance, i.e.,

$$\frac{\partial var(k)}{\partial \beta} = \frac{2(1 - \frac{1}{n})}{(1 + \frac{1}{n} - 2\beta)^2}(k^*)^2 > 0 \tag{48}$$

i.e. increasing the share of profits in national income increases the variance of the steady state distribution. Ultimately, of course, as β creeps upward, when it passes $\frac{1}{2}(1+\frac{1}{n})$ there is no stable variance. This result has a relatively straightforward explanation; according to our assumptions regarding savings behaviour, if individuals did not die they would accumulate infinite wealth; the factor which engenders stability to the distribution is death and the subdivision of wealth amongst n heirs. Increasing the share of profit has the effect of increasing the incomes of those with relatively large amounts of wealth at the expense of those whose incomes are derived principally from earnings. Differential accumulation is thus altered in favour of individuals already high on the wealth scale, and thus, as we should expect, the variance of the distribution will increase.

Consider now the effect of increasing the number of children, first on the coefficient of variation squared,

$$\frac{\partial (c.var.)^2}{\partial n} = \frac{2(1-\beta)}{n^2(1+\frac{1}{n}-2\beta)^2} > 0 \tag{49}$$

i.e. increasing the number of children increases the coefficient of variation. This may appear as a somewhat surprising result, since an increase in wealth splitting may be expected to have the reverse effect. However endogeneity of the wage and interest rate are factors producing this effect. As we see from equations (33), (34) increasing the number of children has the effect of raising the rate of interest and lowering the wage, and from (32) of decreasing mean wealth. Thus although increasing the number of children born to each individual strengthens the stabilizing factor, it also strengthens the dissipative factor by increasing the rate of interest. As (49) shows with respect to the coefficient of variation the dissipative influence is the stronger.

What is the effect of n on the variance of the distribution ?

$$\frac{\partial (var(k))}{\partial n} = \frac{(1-\frac{1}{n})(2k^*)}{(1+\frac{1}{n}-2\beta)}\frac{\partial k^*}{\partial n} + \frac{\partial (c.var.)^2}{\partial n}(k^*)^2 \tag{50}$$

We thus have two contrary influences; the factor working to reduce the variance is the change in mean wealth,

$$\frac{\partial k^*}{\partial n} = \frac{-k^*}{(1-\gamma)(n-1)} < 0 \tag{51}$$

whilst the change in the coefficient of variation will serve to increase the variance. Substituting (49) and (51) in (50) we have,

$$\frac{\partial (var(k))}{\partial n} = \frac{-2(n(1-2\beta)+\beta+\gamma(1-\beta))}{(1+\frac{1}{n}-2\beta)^2(1-\gamma)n^2}(k^*)^2 \tag{52}$$

and thus if $\beta < 0.5$, then certainly,

$$\frac{\partial (var(k))}{\partial n} < 0 \tag{53}$$

4. Wealth Shuffling and Inequality

In the basic model we have assumed that heirs come into their inheritance upon the donor's death; further that such heirs were previously penniless and had not entered the wealth distribution $h(k; t)$. We now wish to consider the more general case in which individuals already occupy a position in the wealth distribution before they inherit; furthermore we should also wish to take account of the fact that wealth transfers may also take place during the lifetime of the donor. In order to consider such possibilities we may modify the basic model by the introduction of a wealth transfer or wealth shuffling function.

Instead of the assumption that individuals leave the distribution with a probability of $\lambda\delta t + o(\delta t)$ within the time interval δt; let us now modify this to the assumption that they simply jump wealth ranges with the same probability. Concerning the length of the jump, we assume this to be defined by the conditional cumulative distribution function,

$$G(k; k^{\star}) \tag{54}$$

which defines the proportion of individuals having terminal wealth less than or equal to k, conditional on initial wealth being $k^{\star}$: The associated frequency distribution function , $g(k; k^{\star})$ defines the proportion of individuals initially having wealth $k^{\star}$ whose terminal wealth is k, i.e.,

$$G(k; k^{\star}) = \int_{x=k_L}^{x=k} g(x; k^{\star})dx \tag{55}$$

where k_L denotes the lower wealth bound.

For those individuals who do not jump we maintain the assumption that they continue to accumulate wealth according to the function $g(:)$. The macroeconomic dimension of the model remains unchanged.

Now consider a small interval of time δt; if the individual's wealth at the end of the period is less than k then this could have come about in one of the following, and mutually exclusive ways. Firstly, the individual's wealth at the beginning of the interval $(t; t + \delta t)$ was less than $k - u_S\delta t$ and there was no jump;secondly, the individual's wealth at the beginning of the period was $k^{\star}$ and a jump equal to or less than $(k - k^{\star})$ has occurred.

We therefore have the following equation for the evolution of the wealth distribution function,

$$F(k; t+\delta t) = (1 - \lambda\delta t)F(k - u_S\delta t; t) + \lambda\delta t \int_{k^{\star}} \frac{\partial F(k^{\star}; t)}{\partial k^{\star}} G(k; k^{\star})dk^{\star} + o(\delta t) \tag{56}$$

i.e. the number of individuals having wealth less than or equal to k at time $(t + \delta t)$ is equal to the numbers of individuals having less than or equal to

$k - u_s\delta t$ at time t, who did not jump; plus the numbers who did jump, these numbers being given by the individuals in various wealth changes, weighted by the proportions in those ranges having a jump to range less than or equal to k at time $(t + \delta t)$.

Expanding $F(k, t + \delta t), F(k - u_s\delta t, t)$ in Taylor series about the point (k, t) then dividing through (56) by δt, and letting $\delta t \to 0$, we arrive at the following integro-differential equation governing the dynamics of the distribution function,

$$\frac{\partial F(k,t)}{\partial t} = -u_s \frac{\partial F(k,t)}{\partial k} - \lambda F(k,t) + \lambda \int_{k^*} \frac{\partial F(k^*,t)}{\partial k^*} G(k,k^*)dk^* \tag{57}$$

The integral in (57) is defined with respect to initial wealth prior to the jump, and the range of integration is over all permissible wealth values. Equation (57) thus takes the pace of equation (2) of the basic model.

In order to arrive at analytical solutions, additional assumptions have to be made with respect to the shuffle function $G(k, k^*)$. We shall assume that the distribution of jumps relative to initial wealth is the same for all values of initial wealth, i.e. $G(k, k^*)$ is of the form $G(k/k^*)$. Differentiating (57) w.r.t k under this assumption, we therefore have the equation,

$$\frac{\partial f(k,t)}{\partial t} = -u_s \frac{\partial f(k,t)}{\partial k} - \left(\frac{\partial u_s}{\partial k} + \lambda\right) f(k,t) + \lambda \int_y f(k/y,t) g(y) dy/y \tag{58}$$

where $y = k/k^*$.

With u_s defined by (1) we may take the moment transform of (58) to give,

$$\frac{dM_f(q,t)}{dt} = spqM_f(q-1,t) + (srq - \lambda + \lambda M_g(q))M_f(q,t) \tag{59}$$

where we define,

$$M_f(q,t) = \int_k k^q f(k,t)dk \tag{60}$$

$$M_g(q) = \int_y y^q g(y)dy \tag{61}$$

i.e. the moment transforms with respect to the distribution of wealth, and the jump function.

Before considering the solution of this model, let us consider the properties of the moment transform of the shuffle function $M_g(q)$. $M_g(0)$ denotes the zero'th moment of the shuffle function; if the numbers being shuffled remain constant then $M_g(0) = 1$. If we wish to introduce the possibility of population growth, then $M_g(0)$ has to exceed unity; hence in order to bring this formulation in line with that of the basic model we shall assume $M_g(0) = n > 1$.

The appropriate value for $M_g(1)$ can be seen from (59) with $q = 1$, i.e.,

$$\frac{dM_f(1,t)}{dt} = spM_f(0,t) + (sr - \lambda + \lambda M_g(1))M_f(1,t) \tag{62}$$

If the shuffle function is truly to shuffle wealth, then wealth cannot be created or destroyed by such a process; thus if an individual transfers wealth to other individuals, the donor's wealth must diminish by the amount of the transfer. Thus in order to maintain this wealth accounting identity, we must have the restriction that $M_g(1) = 1$; as is apparent from (62). No a priori restrictions are placed on the higher moments $M_g(q); q \geq 2$.

Solving the system we note the macro-model gives identical results to the basic model. The normalized moments are given by (35) with the c_j now defined by,

$$c_j = \lambda(n - 1) - (sr^* j - \lambda + \lambda M_g(q)) \tag{63}$$

Provided the stability conditions are satisfied, the steady state values of the higher moments are given by (35) with the d_j now defined by,

$$d_j = 1 - \gamma j + (1 - M_g(j))/(n - 1) \tag{64}$$

and in particular the steady state variance,

$$var(k) = \frac{n - 2 + nm_g(2)}{n - 2\gamma(n - 1) - nm_g(2)}(\bar{k}^*)^2 \tag{65}$$

where $m_g(2)$ denotes the normalized second moment of the shuffle function $M_g(2)/M_g(1)$.

The minimum value of the variance of the shuffle function is, of course, zero; this must therefore imply, at this minimum,

$$m_g(2) = m_g(1)^2 \tag{66}$$

But as we have seen, $m_g(1) = (1/n)$; thus the minimum value of $m_g(2) = (1/n)^2$. Since $\partial var(k)/\partial m_g(2) > 0$; this must therefore imply that the minimum value of the distribution of wealth is reached when $m_g(2) = m_g(1)^2$. Thus we have the conclusion that whatever shuffle function we choose, subject to the constraints noted on the zero'th and first moments, then the steady state variance of the distribution of wealth cannot be below that generated by the basic model.

An alternative viewpoint of the basic model is to view it as the particular case of a shuffle function defined by the Dirac delta function of the form,

$$g(k/k^*) = n\delta(1 - (1/n)) \tag{67}$$

If such a viewpoint is taken, then the basic model can also be interpreted in terms of generalized gifts and inheritances, without necessarily assuming that individuals only enter the distribution on receipt of their inheritance.

5. Inequalities in Savings and Incomes

In the basic model the savings of the individual are known once the wealth of the individual is known. We have further assumed that the wage income and interest rate earned on wealth are identical for all individuals. A closer approximation to reality would allow some dispersion in savings at each level of wealth; such dispersion reflecting inequalities in wage income at each wealth level, differences in the rate of return on wealth, and variations in preferences of individuals regarding choice between consumption and investment.

In relation to the basic model we retain the function form,

$$g(.) = s(p + rk) \tag{68}$$

but now interpret s as the mean savings propensity; and p and r as the mean wage and rate of return received in each wealth range.

We introduce a dispersion factor,

$$u_{ss} = \sigma^2 k^2; \sigma > 0 \tag{69}$$

which describes the variability of savings around mean savings in each wealth range.

The general equation governing the evolution of the distribution of wealth, maintaining the other assumptions of the basic model, can be shown to be,

$$\frac{\partial f(k;t)}{\partial t} = -\frac{\partial}{\partial k}(u_s f(k;t)) + \frac{1}{2}\frac{\partial^2}{\partial k^2}(u_{ss} f(k;t)) - \lambda f(k;t) + \lambda n^2 f(nk;t) \tag{70}$$

Taking the moment transform of (70) and noting that,

$$M(k^2 \frac{\partial^2}{\partial k^2}(f(k;t)) = (q+1)qM(q;t) \tag{71}$$

we have,

$$\frac{dM(q;t)}{dt} = spqM(q-1;t) + (srq - \lambda + \lambda n^{1-q} + \frac{1}{2}\sigma^2 q(q-1))M(q;t) \tag{72}$$

which replaces equation (14) of the basic model; all other moment equations remaining the same. Thus solving the system, we may again note that the macro-model gives identical results to the basic model, consequent on the symmetrical nature of the savings function. All higher moments may be determined, and provided the stability conditions are satisfied, the steady state values are given by (35) with the d_j now,

$$d_j = 1 - \gamma j - (\frac{n^{1-j} - 1}{n-1}) - \frac{1}{2}\frac{\sigma^2 j(j-1))}{\lambda(n-1)} \tag{73}$$

The steady state variance can be shown to be,

$$var(k) = \frac{1 - (1/n) + \frac{\sigma^2}{\lambda(n-1)}}{1 + (1/n) - 2\gamma^*}(k^*)^2 \tag{74}$$

where,

$$\gamma^* = (\gamma + \frac{1}{2}\frac{\sigma^2}{\lambda(n-1)} \tag{75}$$

Thus the implication of introducing a heteroscedastic disturbance term into the specification of savings behaviour may be seen to change the de facto share of income going to profit, when judging the impact on the variance of the distribution of wealth. Such an impact could have been surmised directly from () in which the de facto interest rate can be seen to be raised by the factor $\frac{1}{2}\frac{\sigma^2(q-1)}{s}$. The greater the dispersion in savings the greater the steady state variance of the wealth distribution.

6. Non-Linear Savings Functions

The models considered in previous sections are linear in the sense that the functions describing mean savings behaviour have been linear functions of mean income and thus of wealth. The major implication of this linearity being that the solution for any given moment of the size distribution can be determined recursively in terms of the moments of lower order. As a result, we have seen that the equation for the first moment evolution is synonymous with the familiar macrogrowth equation for capital accumulation, in which the change in aggregate wealth is assumed to be independent of the variance or other higher moments of the wealth distribution.

If non-linearities are introduced into the savings function, then the recursive nature of the moment system may not be retained. For example, if the savings function is quadratic in wealth, i.e.,

$$g(\cdot) = a + bk + ck^2 \tag{76}$$

then the equation for the moments becomes,

$$\frac{dM(q;t)}{dt} + aqM(q-1;t) + bqM(q;t) + cqM(q+1;t) + \lambda(n^{1-q}-1)M(q;t) \tag{77}$$

where we retain the basic model assumptions regarding inheritance. It can be seen that knowledge of at least the variance of the distribution of wealth will be required before we can solve the equation for the evolution of aggregate wealth; whilst knowledge of the evolution of the variance will require knowledge of the third moment, etc. Such systems generate growth models of the economy which are of an exotic nature, and do not form part of the Parthenon of neoclassical growth theory, and therefore are not considered further in this paper. It can of course be argued that the case where $g(\cdot)$ is a function of k to a power greater than unity is not likely to reflect any known cross sectional data, and therefore not likely to be of great practical import; at least where $g(\cdot)$ is required to be a continuous function over all possible values of wealth.

We shall however consider the case where $g(.)$ depends on wealth with an exponent of less than unity; in this case the moment system is solvable by recursive techniques. As a particular example we consider the case where the logarithm of income is assumed to be linearly dependent on the logarithm of the wealth of the individual, with elasticity coefficient $a < 1$, i.e.

$$y = Bk^a \tag{78}$$

Further we assume that savings are proportional to income, and so,

$$g(.) = sBk^a \tag{79}$$

We may note that a savings function of this form was proposed by Clower and Johnson(19**), on the basis of a one period utility maximization model in which utility was a function of both consumption and wealth. The function has also been fitted to Swedish data with some success by Naslund and Sellstedt(19**) the elasticity coefficient turning out to be of the magnitude 0.3.

The equation governing the distribution of wealth becomes,

$$\frac{\partial f(k;t)}{\partial t} = -sBk\frac{\partial}{\partial k}(f(k;t)) - (asBk^{a-1} + \lambda)f(k;t) + \lambda n^2 f(nk;t) \tag{80}$$

where again the inheritance assumptions of the basic model are retained.

Taking the moment transform of (80) we have,

$$\frac{dM(q;t)}{dt} = sBqM(q + a - 1;t) - (\lambda + n^{1-q}))M(q;t) \tag{81}$$

Redefining q as $q = q^*(1 - a)$; (81) may be solved recursively, letting q^* take the values, $q^* = 1, \ldots, N$, to give,

$$\prod_{j=1}^{j=N}(D + \lambda(1 - n^{1-j(1-a)}))M(N(1-a);t) = N!(sB(1-a))^N M(0;t) \tag{82}$$

which may be solved to give, in terms of the normalised "non-integer" moments,

$$m(N(1-a);t) = \frac{M(N(1-a);t)}{M(0;t)} = m(N(1-a);t) = \sum_{j=1}^{j=N} A_j \exp((n^{-j(1-a)} - 1)\lambda t) + \left(\frac{sB(1-a)}{\lambda n}\right)^N N!\left(\prod_{j=1}^{j=N}(1 - n^{-j(1-a)})\right)^{-1} \tag{83}$$

where the A_j are the arbitrary constants determined by the initial distribution of wealth.

Since $n^{-j(1-a)} < 1$ for $j \geq 1$; then $m(N(1-a);t)$ converges to the last term of (83) for all N, and hence we have a limiting stable distribution. Note that for $N = 1$, the equation implicitly defines as a special case the evolution of the Atkinson inequality index (for a given a).

7. Unequal Inheritance and Differential Fertility

In the basic model we assumed that wealth was distributed equally amongst n heirs; furthermore that every individual on death had the same number of heirs. Both of these assumptions may be relaxed.

The assumption that wealth is distributed unequally amongst n heirs can be considered as follows. Let w_i be the share of the ith heir, $\sum_i w_i = 1; 0 \leq w_i \leq 1; all:i$. The equation governing the distribution of wealth is now given by,

$$\frac{\partial f(k;t)}{\partial t} = -s(p+rk)\frac{\partial}{\partial k}(f(k;t)) - (\lambda + sr)f(k;t) + \lambda \sum_{i=1}^{i=n} \frac{1}{w_i} f(\frac{k}{w_i};t) \quad (84)$$

Taking the moment transform we have,

$$\frac{dM(q;t)}{dt} = spqM(q-1;t) + (srq - \lambda + \lambda \sum_{i=1}^{i=n} w_i^q))M(q;t) \quad (85)$$

Thus solving the system we have identical solutions for $q = 0;1$ as the basic model. All higher moments may be determined, and provided the stability conditions are satisfied, the steady state values are given by (35) with the d_j now,

$$d_j = 1 - \gamma j + (1 - \sum_{i=1}^{i=n} w_i^j)/(n-1) \quad (86)$$

In particular, the steady state variance is given by,

$$var(k) = \frac{n - 2 + \sum_{i=1}^{i=n} w_i^2}{n - 2(n-1) - \sum_{i=1}^{i=n} w_i^2}(k^*)^2 \quad (87)$$

Subject to the constraint on $\sum_{i=1}^{i=n} w_i$; $\sum_{i=1}^{i=n} w_i^j$ attains its minimum value when $w_i = (1/n)$; i.e. equal shares. Any movement of shares away from equal subdivision can thus be seen to increase inequality as measured by the variance.

Concerning differential fertility, assume that the population is split into M groups, indexed by m; the proportion of the population in each group is $v_m; 0 \leq v_{m;} \leq 1 \; \sum_m v_m = 1$: Within each group each donor splits wealth equally between n_m heirs.

The equation governing the distribution of wealth is now,

$$\frac{\partial f(k;t)}{\partial t} = -s(p+rk)\frac{\partial}{\partial k}(f(k;t)) - (\lambda + sr)f(k;t) + \lambda \sum_{i=1}^{i=n} v_m n_m^2 f(\frac{k}{w_i};t) \quad (88)$$

Taking the moment transform we have,

$$\frac{dM(q;t)}{dt} = spqM(q-1;t) + \left(srq - \lambda + \lambda \sum_{m=1}^{M} v_m n_m^{1-q}\right)M(q;t) \qquad (89)$$

Solving the system again we note that the macro-model gives identical results to the basic model. Provided the stability conditions are satisfied the steady state values of the higher moments are given by (35) with the d_j now defined by,

$$d_j = 1 - \gamma j + \left(1 - \sum_{m=1}^{N} v_m n_m^{1-j}\right)/(n-1) \qquad (90)$$

In particular, the steady state variance is given by,

$$var(k) = \frac{n - 2 + \sum_{m=1}^{M} v_m n_m^{1-j}}{n - 2\gamma(n-1) - \sum_{m=1}^{M} v_m n_m^{-j}} (k^*)^2 \qquad (91)$$

Letting n denote the average number of heirs over all groups, i.e. $\sum_{m=1}^{M} v_m n_m = n$; then for fixed n, $\sum_{m=1}^{M} v_m n_m^{-1}$ attains its minimum when $n_m = n$ all m. Thus any movement towards differential fertility as between groups can be shown to increase inequality when such is measured by the variance.

8. Individual and Distributional Stability

In the preceding section although there exists the possibility of stability for the distribution, at least for the lower moments, there is no such stability for the individual, who accumulates wealth steadily or dies, in which case the children accumulate wealth starting with their inheritance, until they die, and so on. We shall now change the savings function such that it admits the possibility of a stable equilibrium for the individual.

The savings behaviour we introduce is the proportional savings function modified by the addition of a 'depreciation ' term, i.e.,

$$g(k) = \frac{dk}{dt} = s(p + rk) - \mu k; \quad \mu > 0 \qquad (92)$$

The term 'depreciation' should, of course, not be taken in the literal physical sense. Meade(1964) specifies such a savings function, a priori, arguing that the greater the wealth of the individual the less the individual needs to save, a type of 'precautionary' wealth holding hypothesis. An alternative justification for (92) is that the government has introduced a proportional tax on wealth at the rate μ , the proceeds of which are entirely consumed by the government. This would further entail the assumption that savings from pretax income are totally unaffected by the tax, if s is presumed to take the same value as when $\mu = 0$: Analysis of the effects of taxation will be considered further below. With equation (92) replacing (1) in the basic model, proceeding in the manner as above we can determine the solution for the N th moment as,

$$M(N;t) = \sum_{j=1}^{j=N} C_j \exp((sr - \theta)j - \lambda + \lambda n^{1-j})t$$

$$+ \frac{N!(sp)^N M(0;0)\exp(\lambda(n-1)t)}{\prod_{j=1}^{j=N}(\lambda(n-1) - ((sr-\theta)j - \lambda + \lambda n^{1-j}))} \tag{93}$$

We note that $sr - \theta < 0$, is a necessary condition that all the moments (except, of course, the zero'th, i.e. the population) converge to the second expression of (93), but this condition is not necessary for some of the moments to so converge. The condition $sr - \theta < 0$ also ensures the existence of a stationary value for individual wealth, as may be seen from equation (93). Proceeding in the manner as for the savings equation (1), we may show that the 'Solow' equation regarding the evolution of the first normalized moment may be written as,

$$\frac{dm(1;t)}{dt} = sm(1;t)^{\gamma} - (\theta + \lambda(n-1))m(1;t) \tag{94}$$

Thus again we have stable values of $m(1;t); r(t); p(t)$, which in the limit, as $t \to \infty$, have values given by,

$$m(1;t) = k^{*} = [\frac{s}{\lambda(n-1)+\theta}]^{\frac{1}{1-\gamma}} \tag{95}$$

$$r(t) = r^{*} = \frac{\gamma(\lambda(n-1)+\theta)}{s} \tag{96}$$

$$p(t) = p^{*} = (1-\beta)[\frac{s}{\lambda(n-1)+\theta}]^{\frac{\gamma}{1-\gamma}} \tag{97}$$

Note that it is possible for $rs > \theta$ and yet equations (95)-(97) still hold.

Thus we have the following possibilities,

(i) an equilibrium wealth solution for the individual exists, and thus equilibrium for the normalized distribution.

(ii)no equilibrium for the individual and stable values for the lower moments of the distribution. but not all.

(iii) no equilibrium for the individual or any stable moments.

In the case where individual equilibrium exists, all the normalized moments of the distribution will tend to,

$$m(N;t) = \frac{N!(sp)}{\prod_{j=1}^{j=N}(\lambda(n-1) - ((sr-\theta)j - \lambda + \lambda n^{1-j}))} \tag{98}$$

with p and r determined by (96) and (97).

Of special interest is the stable variance. The condition for a stable second normalised moment is,

$$2(sr - \mu) - \lambda + \frac{\lambda}{n} - \lambda(n - 1) < 0 \tag{99}$$

irrespective of whether $sr \gtrless \mu$, and the variance may be written as,

$$\mathrm{var}(k) = \frac{2(1 - \beta)(k^*)^2}{1 + \frac{1}{n} + \frac{\mu(n-1)}{n(\lambda(n-1)+\mu)} - 2\beta} - (k^*)^2 \tag{100}$$

It may be shown that,

$$\frac{d\mathrm{var}(k)}{dk} < 0 \tag{101}$$

i.e. increasing the 'tax rate' μ reduces the variance of the distribution.

We may note the special case where no individual dies, or what amounts to the same thing that every individual has exactly one heir, i.e. $\lambda = 0$; or $n = 1$. In this case we would expect all individuals whatever their initial wealth to converge to the stationary point of wealth implied by equation (95), provided $sr < \mu$. The distribution function should thus become a Dirac delta function at this point. Does our formal analysis confirm this intuitive result ?

Putting $n = 1$ in (98) we have,

$$m(N;t) = \frac{N!(sp)^N}{\prod_{j=1}^{j=N}(\mu - sr)j} = \left[\frac{sp}{(\mu - sr)}\right]^N \tag{102}$$

Now note the following result, if,

$$h^*(k;t) = \delta(k - k_0) \tag{103}$$

where $\delta(.)$ is the Dirac delta function and $h(k;t)$ the normalized distribution; then,

$$M(\delta(k - k_0); N) = \int_{k=0}^{k=\infty} k^N \delta(k - k_0)dk = k_0^N \tag{104}$$

Thus,

$$M^{-1}(m(N;t)) = \delta\left(k - \frac{sp}{(\mu - sr)}\right) \tag{105}$$

i.e. the distribution tends to a Dirac delta function at the point,

$$k = \frac{sp}{(\mu - sr)} \tag{106}$$

and this is the stationary point implied by equation (102).

9. Alternative Production and Savings Functions.

There are a number of further generalizations to the basic linear model which may be considered. These include alternative specifications of the production function, including technical progress;changes in the form of the consumption function, e.g. savings propensities dependent on the income component; taxation of income and wealth. Again the method of solution is as above; first derive the time paths of the macroeconomic parameters, and then if limiting values of these parameters exist, substitute them into the equation governing the evolution of the distribution function, or its moment transform and solve. We shall consider the following cases:

(i) firstly the solution for the general linear savings function and a production function satisfying the Inada conditions:

(ii)secondly, the solution for the Harrod-Domar production function; and,

(iii) the solution for a 'Kaldorian' savings function, and fourthly,

(iv) the solution for the AK production model.

We treat these problems rather briefly; they may be viewed as just formal exercises once the basic model has been understood.

(i) General Linear Savings Function and Inada Production Function

We assume that the equation governing the individual's wealth is,

$$\frac{dk}{dt} = s(p + rk) - \mu k - d \tag{107}$$

and the aggregate production function,

$$Y(t) = F(K(t); L(t)) \tag{108}$$

such that,

$$\bar{y}(t) = \frac{Y(t)}{L(t)} = F(\frac{K(t)}{L(t)}; 1) \equiv f(k) \tag{109}$$

where $f(:)$ satisfies the usual Inada conditions, $f(k) > 0; f'(k) > 0; f''(k) < 0; f(0) = 0; f(\infty); f'(0) = \infty; f'(\infty) = 0; k \geq 0$:

The equation governing the evolution of the moments of the distribution function can be derived as,

$$\frac{dM(N; t)}{dt} = ((sr - \mu)N - \lambda + \lambda n^{1-N})M(N; t) + N(sp - d)M(N - 1; t) \tag{110}$$

First we require knowledge of whether stable values of r and p exist. The 'Solow' equation for the evolution of the normalized first moment can be derived as,

$$\frac{dm(1; t)}{dt} = s\bar{y}(t) - (\mu + \lambda(n - 1))m(1; t) - d \tag{111}$$

In Fig.1 we have illustrated the case where two stationary normalized first moments exist at $m^*(1)$ and $m^{**}(1)$:

(Fig.1 to appear here)

Movement towards m^{**} or away from m^{*} depends on the normalized first moment of the initial distribution of wealth. We may solve equation (111) assuming that we have stationary values of r and p, but not yet specifying which set.

The solution for the moments is,

$$M(N;t) = \sum_{j=1}^{j=N} C_j \exp((sr - \theta)j - \lambda + \lambda n^{1-j})t$$

$$+ \frac{N!(sp - d)^N M(0;0)\exp(\lambda(n-1)t)}{\prod_{j=1}^{j=N}(\lambda(n-1) - ((sr-\theta)j - \lambda + \lambda n^{1-j}))} \qquad (112)$$

If we assume that the factor prices are determined by marginal productivity criteria then the conditions for convergence of the moments at either of the two stationary points are straightforward. From Fig.1 we note that,

$$\text{at } m^{*}(1) \text{ we have } sr > \theta + \lambda(n-1) \qquad (113)$$

$$\text{at } m^{**}(1) \text{ we have } sr < \theta + \lambda(n-1) \qquad (114)$$

Thus at the lower equilibrium,

$$(sr^{*} - \theta)j > j\lambda n - j\lambda > j\lambda n > \lambda > 0 \qquad (115)$$

since j and n are both greater than one.

Thus all moments, including the first,diverge, as do the normalized moments. The condition for convergence of the normalized moments is,

$$(sr - \theta)j - \lambda - \lambda(n-1) + \lambda(n-1) + \lambda n^{1-j} < 0 \qquad (116)$$

Thus if $(sr - \theta)j > \lambda n$ the moments certainly diverge, and this is the case at the lower equilibrium as we see from (74).

At the upper equilibrium the first normalized moment converges since $(sr^{**} - \theta) < \lambda(n-1)$: However since,

$$(sr^{**} - \theta)j \gtrless \lambda n \qquad (117)$$

the higher moments may or may not diverge. Certainly, all higher normalized moments will converge if

$$sr^{**} < \theta \qquad (118)$$

which is also the condition for a stationary equilibrium for individual wealth to exist,. The qualitative behaviour of the solution at the upper equilibrium for $m(1;t)$ is thus similar to the solution of the basic model in the previous section.

(b) Harrod-Domar Production Function

We assume the aggregate production function to be given by,

$$\frac{Y(t)}{L(t)} = \frac{1}{v}\frac{K(t)}{L(t)} \tag{119}$$

In this case, given that the share of profit is β, the rate of interest that each individual will receive on his wealth is,

$$r(t) = \frac{\beta}{v} \tag{120}$$

and the wage that each individual will receive is

$$p(t) = \frac{1-\beta}{v}\frac{K(t)}{L(t)} \tag{121}$$

Assuming that individuals save a constant proportion of their income, the change in the individual's wealth is given by,

$$\frac{dk}{dt} = s(p + rk) = \frac{s}{v}((1-\beta)\frac{K(t)}{L(t)} + \beta k(t)) \tag{122}$$

Note that $\frac{K(t)}{L(t)}$ denotes the per capita capital whilst k(t) denotes the wealth of the individual.

The equation for the moments of the distribution h(k,t) may be derived as,

$$\frac{dM(N;t)}{dt} = (N\frac{s\beta}{v} - \lambda + \lambda n^{1-N})M(N;t) + (1-\beta)\frac{s}{v}\frac{K(t)}{L(t)}N M(N-1;t) \tag{123}$$

The solutions for the zero'th and first moments, pitting $N = 0$, and $N = 1$ in (123) are,

$$L(t) = L(0)e^{\lambda(n-1)t} \tag{124}$$

and,

$$K(t) = K(0)e^{(s/v)t} \tag{125}$$

Thus,

$$\tilde{k}(t) = \frac{K(t)}{L(t)} = \frac{K(0)}{L(0)}\exp(\frac{s}{v} - \lambda(n-1))t \tag{126}$$

In balanced growth, $K(t)/L(t) = \tilde{k}^*$; i.e. when $(s/v) = \lambda(n-1)$; the well known 'knife edge' property of the Harrod-Domar model. Assuming that the normalized moments up to a given order are in equilibrium, we may solve for the immediately succeeding moment in terms if the lower. The equation for the normalized moments of order greater than or equal to two is given by,

$$\begin{aligned}\frac{dm(N;t)}{dt} &= ((\beta N - 1)\lambda(n-1) - \lambda + \lambda n^{1-N})m(N;t) \\ &\quad + (1-\beta)\lambda(n-1)\tilde{k}^* N m(N-1;t)\end{aligned} \tag{127}$$

where we have assumed the first normalized moment to be in balanced growth and thus $(s/v) = \lambda(n-1)$. Whether the N'th moment has an equilibrium value depends on whether,

$$(\beta N - 1)\lambda(n-1) - \lambda + \lambda n^{1-N} \gtrless 0 \tag{128}$$

Ultimately, of course, this expression is bound to become positive as N increases. Naturally, if the first normalised moment is not in equilibrium, i.e. we are not on the 'knife edge', then none of the higher moments have equilibrium values either.

(c) The Kaldorian Savings Function

The case where different proportions are saved according to the nature of the income course can easily be considered. We take the basic model with (10) replaced by,

$$\frac{dk}{dt} = s_p p + s_r r k \tag{129}$$

where s_p denotes the propensity to save from the earned income component, and s_r the savings propensity for the unearned component, $0 < s_p < s_r < 1$.

It may appear at first sight to be rather unrealistic to assume that an individual saves different proportions from different income sources; however it has some justification if we remember that () refers to the mean savings in any wealth state. All that () then asserts is that if $s_r > s_p$ then the greater the individual's wealth, the greater will be the proportion of income saved. This of course runs counter to Meade's suggestion noted above.

The equation for the moments becomes,

$$\frac{dM(N;t)}{dt} = ((s_r r)N - \lambda + \lambda n^{1-N})M(N;t) + N(s_p p)M(N-1;t) \tag{130}$$

The equation for the evolution of the normalized first moment is,

$$\frac{dm(1;t)}{dt} = (s_r\beta + s_p(1-\beta))m(1;t)^{\gamma} - \lambda(n-1)m(1;t) \tag{131}$$

and this has a steady state value given by,

$$k^{*} = \left[\frac{s_r\beta + s_p(1-\beta)}{\lambda(n-1)}\right]^{\frac{1}{1-\gamma}} \tag{132}$$

the corresponding values of r and p at equilibrium, are given by () and () with $s = s_r\beta + s_p(1-\beta)$.

The solution of () for the normalized moments is,

$$m(N;t) = \frac{M(N;t)}{M(0;t)} = \sum_{j=1}^{j=N} C_j \exp(s_r r j - \lambda + \lambda n^{1-j} - \lambda(n-1))t$$

$$+\frac{N!(s_p p)^N}{\prod_{j=1}^{j=N}(\lambda(n-1)-(s_r r j-\lambda+\lambda n^{1-j}))} \tag{133}$$

Note that here, in distinction to the basic model, the share of profit does determine the mean wealth ratio in equilibrium. It is only the savings propensity of unearned income which influences the convergence of moments higher than the first.

(d) The AK production function
(to be completed)

10. Taxation and the Distribution of Wealth

In this section we propose to consider the manner in which different taxes affect the inequality in the wealth distribution as measured by the variance, and their affects on the total accumulation of capital. Taxes will be introduced into the basic model specified above. The government is viewed purely as a passive instrument; it levies taxes and redistributes the proceeds back to the individuals in the form of welfare payments.

The taxes we consider are:-

(i) A Tax on Earned Income, (ii) A Tax on Unearned Income, (iii) A Tax on Wealth, and (iv) A Tax on Inheritance.

All proceeds of taxation will be assumed to be distributed equally amongst the population.

(i),(ii),(iii) A Proportional Tax on Earned Income, Unearned Income, and Wealth

Considering the representative individual, income before tax is,

$$y = p + rk \tag{134}$$

After tax and before welfare payments it is,

$$y_1 = (1-t_p)p + (1-t_r)rk - t_k k \tag{135}$$

where $t_p; t_r;$ and t_k are the proportional tax rates on earned income, unearned income and wealth respectively. The government receipts from all taxes at time t we denote by $T(t)$; therefore each individual receives in welfare payments the sum $T(t)/L(t)$. We assume that a constant proportion of disposable income is saved, then,

$$\frac{dk}{dt} = sy_d = s((1-t_p)p + (1-t_r)rk + T(t)/L(t) - t_k k) \tag{136}$$

Note, we have assumed that the individual views the tax on wealth as a component of disposable income.

To determine the total tax receipts $T(t)$ we simply sum all the individual tax payments. The amount that one individual pays in tax is,

$$t_p p + t_r r k + t_k k \tag{137}$$

Therefore total tax receipts are,

$$T(t) = \int_{k=0}^{k=1} (t_p p + t_r r k + t_k k) h(k;t) dk = t_p p M(0;t) + (t_r r + t_k) M(1;t) \tag{138}$$

where $M(:;t)$ denotes the Mellin transform as defined above.

Substituting (138) in (136) we have,

$$\frac{dk}{dt} = s y_d = s((1 - t_p)p + (1 - t_r) r k - t_k k + t_p p + (t_r r + t_k)\frac{M(1;t)}{M(0;t)}) \tag{139}$$

As expected, the tax on earned income cancels out, since with a perfectly equal distribution of earned income, and equal welfare payments, each individual gets back a component of welfare payment exactly equal to the earned income tax component.

The main equations of the model are therefore,

$$\frac{\partial h(k;t)}{\partial t} = -g(k;r;p;taxes)\frac{\partial h(k;t)}{\partial k} - (g_k + \lambda) h(k;t) + \lambda n^2 h(nk;t) \tag{140}$$

where,

$$g(:) = \frac{dk}{dt} = s(p + (r - t_r r - t_k)k + (t_r r + t_k)\frac{M(1;t)}{M(0;t)}) \tag{141}$$

Substituting (141) in (140), and then taking the Mellin transform, we may derive,

$$\frac{dM(N;t)}{dt} = (s(r - t_r r - t_k)N - \lambda + \lambda n^{1-s}) M(N;t) + s(p + (t_t r + t_k)\frac{M(1;t)}{M(0;t)}) N M(n-1;t) \tag{142}$$

When $N = 0$,

$$\frac{dM(0;t)}{dt} = \lambda(n - 1) M(0;t) \tag{143}$$

Hence,

$$M(0;t) = M(0;0) e^{\lambda(n-1)t} \tag{144}$$

When, $N = 1$,

$$\frac{dM(1;t)}{dt} = s r M(1;t) + s p M(0;t) = s Y(t) \tag{145}$$

with the corresponding equation for the normalised first moment as,

$$\frac{dm(1;t)}{dt} = s\bar{y}(t) - \lambda(n - 1) m(1;t) \tag{146}$$

This might appear, at first sight, a rather surprising result, i.e. that aggregate wealth accumulation is unaffected by the levy of our specified taxes. This would appear perfectly understandable in the case of the income taxes/welfare system; since with a proportional savings function, shifting the income distribution by taxes/transfers would make no difference to aggregate savings. In the case of the wealth tax we might presume there to be an effect of switching wealth from rich to poor, wealth that would otherwise not be consumed.

The reason for this outcome lies in our assumption that individuals treat the wealth tax as a component of income ; if they do not then it can be shown that aggregate accumulation is affected.

Assuming a Cobb-Douglas production function the stable values form $(1;t)$, p and r are again given by equations (26) to (28). Since we know that a steady state exists we may thence solve for the higher moments as,

$$M(N;t) = \sum_{j=1}^{j=N} C_j \exp(s(r - t_r r - t_k)j - \lambda + \lambda n^{1-j}) + \frac{N! s^N ((p + (t_r r + t_k)\frac{M(1;t)}{M(0;t)})^N M(0;0) e^{\lambda(n-1)t}}{\prod_{j=1}^{j=N}(\lambda(n-1) - (s(r - t_r r - t_k)j - \lambda + \lambda n^{1-j}))} \tag{147}$$

Assuming stability for the normalized second moment we have,

$$m(2;t) = \frac{2s^2(p + (t_r + t_k/r)r(\frac{M(1;t)}{M(0;t)})^2}{(\lambda(n-1) - s(r - t_r r - t_k))(\lambda(n-1) - 2s(r - t_r r - t_k) + \lambda + \lambda/n)}$$

and this may be expressed as,

$$m(2;t) = \frac{2(1-\beta^*)^2(\bar{k}^*)^2}{(1-\beta^*)(1-2\beta^*+1/n)} \tag{148}$$

where,

$$\beta^* = \beta(1 - t_r - t_k/r) \tag{149}$$

Since we are in equilibrium, β^* is constant, thus

$$var(k) = \frac{(1-1/n)(\bar{k}^*)^2}{(1+1/n-2\beta^*)} \tag{150}$$

and,

$$\frac{\partial var(k)}{\partial \beta^*} = \frac{2(1-1/n)(\bar{k}^*)^2}{(1+1/n-2\beta^*)^2} > 0 \tag{151}$$

and since,

$$\frac{\partial \beta^*}{\partial t_r} = -\beta < 0 \tag{152}$$

$$\frac{\partial \beta^*}{\partial t_k} = -\frac{\beta}{r} < 0$$

Then,

$$\frac{\partial var(k)}{\partial t_r} < 0 \quad (153)$$

$$\frac{\partial var(k)}{\partial t_k} < 0 \quad (154)$$

i.e. increasing both the wealth tax and the tax on unearned income have the effect of reducing the 'de facto' share of profit β^* and thus of reducing the variance of the distribution of wealth.

(iv) The Inheritance Tax

We now consider the effects on the distribution of wealth of a proportional inheritance tax levied by the government on the property owned by all individuals who die. The proportion of wealth paid in tax is assumed to be t_d; so an individual who before death owns wealth to the value $k/(1-t_d)$ would actually pass on to a single child the sum k; thus an individual with wealth $nk/(1-t_d)$ would pass on to each of n children, assuming inheritances were divided equally, the amount k.

Now consider the cumulative distribution function of wealth,

$$F\left(\frac{nk}{(1-t_d)}\right) \quad (155)$$

this gives the number of individuals who have wealth less than or equal to $nk/(1-t_d)$;now if any of these individuals died, we assume that their wealth would be divided equally amongst n heirs, and each of these heirs would thus enter the distribution having wealth less than or equal to k. Thus the equation governing the evolution of the cumulative distribution function is,

$$\frac{\partial F(k;t)}{\partial t} = -g\frac{\partial F(k;t)}{\partial t} - \lambda F(k;t) + \lambda n F\left(\frac{nk}{1-t_d};t\right) \quad (156)$$

and the equation governing the evolution of the density distribution function is thus,

$$\frac{\partial h(k;t)}{\partial t} = -g\frac{\partial h(k;t)}{\partial t} - (g_k + \lambda)F(k;t) + \lambda\frac{n^2}{1-t_d}h\left(\frac{nk}{1-t_d};t\right) \quad (157)$$

We now have to determine the total receipts of the tax; the receipts are then presumed to be distributed equally amongst all individuals. For each individual who dies the tax collected is t_d times wealth. The proportion of individuals dying in each wealth range is λ, thus in each wealth range the total wealth liable to taxation is equal to the numbers dying in that range times the wealth of that range, i.e. $\lambda h(k;t)k$; total tax paid in each range is thus $t_d\lambda h(k;t)k$; and total receipts of this tax for the whole economy is thus the sum of tax payments over all ranges. Letting this amount be $T(t)$, we thus have,

$$T(t) = \lambda \int_{k=0}^{k=1} t_d k h(k;t) dk = \lambda t_d M(1;t) \tag{158}$$

where $M(:;t)$ denotes the Mellin transform of $h(k;t)$ as defined above. Since this sum is distributed equally across the population, each individual receives in welfare payments the amount,

$$\lambda t_d \frac{M(1;t)}{M(0;t)} \tag{159}$$

The total disposable income of the representative individual is thus,

$$y_d(t) = p + rk(t) + \lambda t_d \frac{M(1;t)}{M(0;t)} \tag{160}$$

and assuming that we have a proportional savings function,

$$\frac{dk}{dt} = g(k) = s(p + rk(t) + \lambda t_d \frac{M(1;t)}{M(0;t)}) \tag{161}$$

The equation governing the evolution of the distribution function is,

$$\begin{aligned} \frac{\partial h(k;t)}{\partial t} = & -(s(p + rk + \lambda t_d m(1;t))\frac{\partial h(k;t)}{\partial t} - (sr + \lambda)F(k;t) \\ & + \lambda \frac{n^2}{1 - t_d} h(\frac{nk}{1 - t_d}; t) \end{aligned} \tag{162}$$

Now taking the Mellin transform of (162), we ,may derive

$$\frac{dM(N;t)}{dt} = (srN - \lambda + \lambda n^{1-N}(1 - t_d)^N) M(N;t) + s(p + \lambda t_d m(1;t)) N M(n - 1;t) \tag{163}$$

Thus the equation for the evolution of the normalized first moment is,

$$\frac{dm(1;t)}{dt} = s\frac{Y(t)}{M(0;t)} - (\lambda(n - 1) + (1 - s)\lambda t_d m(1;t) \tag{164}$$

The rate of inheritance tax does affect aggregate capital accumulation in this model;the higher this tax rate the lower the rate of capital accumulation. This results from the fact that the inheritance tax is not viewed as a component of income by the individuals from whom it is taken (they are dead at this time) and during their lifetime no compensating change in savings has occurred. When the tax is taken and distributed as welfare payments, a part of the capital stock is thus consumed which would otherwise not occur.

Assuming a Cobb-Douglas production function, then stable values for $\hat{k}(t), p(t), r(t)$, are given by,

$$k(t)^{1-\gamma} = \frac{s}{\lambda(n-1) + (1-s)\lambda t_d} = (\hat{k}^*)^{1-\gamma} \tag{165}$$

$$r(t) = \frac{\beta}{s}(\lambda(n-1) + (1-s)\lambda t_d) = r^* \tag{166}$$

$$p(t) = (1-\beta)(\hat{k}^*)^{\gamma} = p^* \tag{167}$$

and thus the higher the tax rate t_d; the lower the steady state capital-labour ratio, the higher the rate of interest, and the lower the wage.

Since we know that a steady state value for $m(1;t)$ exists, we may substitute this value, and the corresponding $p(t)$ and $r(t)$ into (163) and solve recursively for the higher moments to give,

$$\begin{aligned} M(N;t) &= \sum_{j=1}^{j=N} C_j \exp((srj - \lambda + \lambda(\frac{n}{1-t_d})^{1-j}(1-t_d))t) \\ &+ \frac{N!(s+\lambda t_d \hat{k}^*)^N M(0;0) e^{\lambda(n-1)t}}{\prod_{j=1}^{j=N}[\lambda(n-1) - (srj - \lambda + \lambda(\frac{n}{1-t_d})^{1-j}(1-t_d))]} \end{aligned} \tag{168}$$

for $N \geq 2$.

Assuming stability, the normalised second moment in equilibrium therefore takes the value,

$$m(2;t) = \frac{2s^2(p+\lambda t_d \hat{k}^*)^2}{(\lambda(n-1) - sr + \lambda t_d)(\lambda(n-1) - 2sr + \lambda - (\frac{n}{1-t_d})^{1-j}(1-t_d)} \tag{169}$$

which on substituting for p and r from (129),(130) may be written,

$$m(2;t) = \frac{2(1-\beta^{**})(\hat{k}^*)^2}{(1-2\beta^{**} + \frac{1}{n}(1-t_d))} \tag{170}$$

where,

$$\beta^{**} = (1 - \frac{st_d}{(n-1+t_d)})\beta \tag{171}$$

Then,

$$\mathrm{var}(k) = m(2;t) - (m(1;t))^2 = \frac{1-\frac{1}{n}(1-t_d)}{(1-2\beta^{**} + \frac{1}{n}(1-t_d))}(\hat{k}^*)^2 \tag{172}$$

in equilibrium. Note that if $t_d = 0; \beta^{**} = \beta$.

We may note that $\frac{\partial \beta^{**}}{\partial t_d} < 0$; i.e. an increase in the inheritance tax, provided the receipts are divided equally amongst the population, will reduce the 'de facto' share of profit. Letting, $(c.var.)^2 = var(k)/(\bar{k}^*)^2$, then we may show,

$$\frac{\partial}{\partial t_d}(c.var)^2 = \frac{2}{n(1 - 2\beta^{**} + \frac{1}{n}(1 - t_d))^2}\left(1 - \beta\frac{(n-1)(1+s) + t_d(1-s)}{n - 1 + t_d}\right) \quad (173)$$

and thus, we may have,

$\frac{\partial}{\partial t_d}(c.var.)^2 < 0$, only if $\beta\frac{(n-1)(1+s)+t_d(1-s)}{n-1+t_d}) > 1$.

The effect of a change in t_d on the coefficient of variation may in fact be more clearly understood if we break it down into two separate effects; firstly ,just the imposition of the tax, and secondly, the effects of distributing the proceeds amongst the population.

Considering first the 'pure tax effect'; we may assume that the government consumes all the tax receipts, or alternatively that the individuals consume all their welfare payments. In this case the equation governing the moments of the wealth distribution is,

$$\frac{dM(N;t)}{dt} = (srN - \lambda + \lambda n^{1-N}(1 - t_d)^N)M(N;t) + spNM(n - 1;t) \quad (174)$$

and thus we have the steady state values,

$$(\bar{k}^*)^{1-\gamma} = \frac{s}{\lambda(n - 1) + \lambda t_d} \quad (175)$$

$$r^* = \frac{\beta}{s}(\lambda(n - 1) + \lambda t_d) \quad (176)$$

$$p^* = (1 - \beta)(\bar{k}^*)^\gamma \quad (177)$$

$$var(k) = \frac{1 - \frac{1}{n}(1 - t_d)}{1 - 2\beta + \frac{1}{n}(1 - t_d)}(\bar{k}^*)^2 \quad (178)$$

Thus the 'pure tax effect' of an increase in t_d. Now introducing the welfare payments effect, we have,

$$\frac{d}{dt_d}(c.var.)^2 = \frac{2(1 - \frac{(1-t_d)}{n})}{(1 - 2\beta^{**} + \frac{1}{n}(1 - t_d))^2}\frac{\partial \beta^{**}}{\partial t_d} + \frac{\partial}{\partial t_d}(c.var.)^2 \quad (179)$$

and thus since $\partial\beta/\partial t_d < 0$, the welfare payments effect works in the contrary direction to the pure tax effect, serving to reduce the coefficient of variation; the overall change in thus being ambiguous.

11. Conclusion

(to complete)

Mathematical Appendix

(to complete)