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The Dynamics of the Wealth Distribution and the Interest Rate with Credit Rationing

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With decreasing returns and first-best credit, the long-run interest rate and aggregate output are uniquely determined, and wealth dispersion among individuals or firms is irrelevant. Introducing credit rationing into the Solow model modifies these conclusions. Multiple stationary interest rates and wealth distributions can exist because higher initial rates can be self-reinforcing through higher credit rationing and lower capital accumulation. The wealth accumulation process is ergodic in every steady state, but wealth mobility is lower with higher steady-state interest rates. Aggregate output is higher in steady states with lower interest rates because credit is better allocated. Short-run interest rate or distribution shocks can be self-sustaining and can have long-run effects on output through the induced dynamics of the wealth distribution and credit rationing.

1. INTRODUCTION

In Solow's model of capital accumulation, the equilibrium interest rate is determined by the marginal product of capital which is common across all agents. A consequence is the irrelevance of the wealth distribution: long-run capital stock, output and interest rates are all uniquely determined by savings behaviour *independently* of the initial wealth distribution. When credit markets are less than perfect, the situation may change: frictions in the credit market may lead to credit-rationing and upset the simple relationship between marginal product of capital and interest rates. The main aim of this paper is to explore a consequence of this: that there may be some persistency in the dual dynamics of the interest rate and the wealth distribution. It becomes possible that both high and low interest rates are self-sustaining. Higher interest rates induce a higher steady-state fraction of credit-constrained individuals, and therefore lower long-run capital accumulation. Our investigation indicates that this steady-state multiplicity is likely to occur when credit constraints become sufficiently tight at high interest rates that it takes a long time for credit-constrained individuals to rebuild their capital.

To each stationary interest rate, there is associated a unique stationary wealth distribution. Each of these stationary distributions is shown to be ergodic, so a poor dynasty has a positive probability of becoming rich in finite time and vice-versa: there is no inescapable poverty trap. However, the degree of wealth and income mobility do vary across steady states. Both upward and downward mobility are greater when the interest rate is lower, so some steady states have *more* of a "poverty trap" than others. Steady states can also be ranked in terms of aggregate output: higher steady-state interest rates are associated with lower output and capital stock, because they involve a higher fraction of credit-constrained individuals who invest and accumulate at inefficiently low levels. Evidently, there is a possible role for policy. A one-off lump sum manipulation of the wealth distribution or the interest rate might lead the economy to a different steady-state than would otherwise occur.

Galor and Zeira (1993) have already pointed out that persistency phenomena can easily arise when there is credit rationing. However, their mechanism is different from ours in that it relies entirely on a non-convex technology. They show that with a fixed-size investment technology poverty traps can occur with an exogeneous interest rate. Poor agents are unable to afford the fixed-size investment which would enable them to accumulate enough to pass the poverty trap threshold. So, initial poverty persists in the long run in the sense that some dynasties never get rich.[1] This is in contrast to the ergodicity of steady states in our model: our multiplicity is based upon the interaction between the wealth distribution and the equilibrium interest rate and not upon threshold effects in wealth accumulation.

Unlike Galor and Zeira, Aghion and Bolton (1997) introduce the interaction between-the wealth distribution and the equilibrium interest rate, and our model is directly inspired from theirs. Aghion and Bolton focus on finding conditions under which there is a non-monotonic evolution of income inequality towards a *unique* steady state. So, their results are very different from ours, and both papers should be viewed as complementary.

Banerjee and Newman (1993) have already recognized a source of multiplicity similar to that studied in the present paper, but they study the dual dynamics of the wealth distribution and *wage rates* with the rate of interest fixed exogenously.[2] In their model, low wage rates slow down the accumulation of the poor and therefore preserves the high initial supply of monitored labour, which in turn reinforces the low equilibrium wage rates (and conversely for the high-wage-rate steady state). The intuition for our results is essentially identical, with the capital market instead of the labour market and high (resp. low) interest rates instead of low (resp. high) wage rates. This suggests the robustness of this type of results: with wealth effects, both the long-run wage rate and the long-run interest rate are likely to vary with the initial wealth distribution.

Our motivation for the formulation chosen in the present paper is that it is designed to be as close as possible to the Solow model. The three papers mentioned above all assume a fixed-size investment technology, whereas we assume a standard concave production function $f(k)$, which allows the effects of introducing credit market imperfections to be separated out from ancillary assumptions such as the introduction of non-convex technologies.

Section 2 sets the scene by analysing the model without credit rationing. Section 3 introduces credit-rationing and derives its static properties. The main results of the paper are presented in Section 4 where the persistence phenomena described above are analysed. Section 5 concludes. An appendix contains all omitted proofs.

2. DYNAMICS WITH FIRST-BEST CREDIT

This section presents the Solow-type capital accumulation model that will be used throughout the paper, and summarizes the main properties of the dynamics of the

1. Note that any small "convexification" of Galor and Zerira's dynamic income process (for example by introducing ε probabilities of moving back and forth the poverty trap threshold) would imply that the system converges toward a unique stationary distribution, irrespective of the initial distribution.

2. In our Solow-type model, every agent is an entrepreneur and the "wage rate" is simply equal to individual output minus interest payments. Banerjee and Newman (1993) obtain a richer occupational structure and two independent market prices for capital and labour by assuming that effort incentives can be dealt with either through financial contracts and the credit market (as in our paper) or by direct monitoring by an entrepreneur (monitored agents then become wage-earners and are priced on a market for monitored labour).

wealth distribution and the interest rate implied by the assumption of first-best credit.[3]

We consider a closed economy with an infinite, discrete time horizon $t=0, 1, 2, \ldots$ and a stationary population of infinitely-lived dynasties $I=[0, 1]$. There are two goods, one labour good and one physical good that can serve both as a consumption good and a capital good. At each period t the state of the economy is described by the current distribution of wealth, represented by a distribution functions $G_t(w)(G_t(w)$ is the fraction of the population with current wealth below w). Aggregate wealth (which is also the average wealth) W_t is given by

$$W_t = \int w dG_t(w).$$

At each period t, every dynasty $i \in I$ endowed with one indivisible labour unit and an initial wealth w_{it}, and earns income by supplying labour and capital; the resulting income y_{it} is divided at the end of the period between consumption c_{it} and savings b_{it}, which are to constitute the dynasty's initial wealth next period (i.e., $w_{it+1}=b_{it}$).

Agents are assumed to be risk neutral: they maximize total expected income minus the disutility of labour, i.e. $U=y-e$, where $e=0$ or 1 is labour supply (effort).[4] Following Solow and the recent literature on distributional dynamics with credit-rationing, we assume that a fixed fraction s of total income is being saved ($b_{it}=sy_{it}$): if we interpret each time period as exactly one generation of each dynasty, one can think of each dynasty as maximizing Cobb–Douglas preferences defined directly over consumption and bequest (say that each generation is maximizing $U=zc^{1-s}b^s-e$, with $z=(1-s)^{s-1}s^{-s}$, so that indirect utility for income is simply $U=y-e$, and $c=(1-s)y$, $b=sy$).

We also assume that wealth can be stored costlessly, but that capital investments are sunk costs (i.e., a 100% depreciation rate).[6]

The technology $F(K, L)$ exhibits constant-returns-to-scale with respect to aggregate capital and labour inputs K and L; in the usual way, we can study production at the individual level, viewing each agent as a prospective entrepreneur; the production function can be written $f^*(k)=F(K/L, 1)$ (with $k=K/L$) at the *per capita* level. The only difference with the usual neo-classical production is that we allow it to be stochastic at the individual level: $f^*(k)$ can take different values depending on purely idiosyncratic shocks (which cancel out at the aggregate level since we have a continuum of agents). To fix ideas, we assume that $f^*(k)$ can take two values:

$$f^*(k)=\begin{cases} f(d) & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases} \quad \text{if individual effect } e=1$$

and

$$f^*(k)=\begin{cases} f(k) & \text{with probability } q \\ 0 & \text{with probability } 1-q \end{cases} \quad \text{if individual effort } e=0,$$

3. See Stiglitz (1969) for a more detailed analysis of the dynamics of the distribution of income and wealth among individuals in the Solow framework with first-best credit.

4. Positive risk-aversion would not complicate the analysis providing no verifiability problem prevents individuals from obtaining full insurance: this would just make individual incomes deterministic, which makes global convergence even more extreme (see below). However when we introduce verifiability problems (as we do in Sections 3 and 4 below), risk aversion would slightly obscure the analysis because credit-rationing issues would be mixed up with partial-insurance issues. Assuming risk neutrality makes more transparent the role of credit-rationing per se (see the working paper version (Piketty (1992)) for a more general analysis with risk-aversion and partial insurance issues).

5. See, e.g., Banerjee and Newman (1993), Galor and Zeira (1993) and Aghion and Bolton (1997).

6. Minor notational changes in what follows would accommodate alternative assumptions.

with $0<q<p<1$ and standard properties for $f(k)$: $f(0)=0$, $f'>0$, $f''<0$, $f'(0)=\infty$, $f'(\infty)=0$.

The role of the shirking option ($e=0$) will become crucial when we introduce incentive problems and credit rationing in subsequent sections, and so will the value of q. In this section however, we consider thecase of first-best credit, which means that there is no moral hazard problem, i.e. that lenders can make sure at no cost that borrowers don't shirk and do supply their unit of effort once the loan has been made. So in this section all individuals always supply a high effort level $e=1$, providing obviously that this is indeed the first-best optimum. For any $r\geqq 0$, we note $k(r)$ (resp. $k_0(r)$) and $y(r)$ (resp. $y_0(r)$) the profit-maximizing capital input and the corresponding profit when the interest rate is r and the entrepreneur takes effort $e=1$ (resp. $e=0$):

$$\forall r\geqq 0, pf'(k(r))=1+r \quad \text{and} \quad y(r)=pf(k(r))-(1+r)k(r)$$

$$qf'(k_0(r))=1+r \quad \text{and} \quad y_0(r)=qf(k_0(r))-(1+r)k_0(r).$$

We then assume that at least when the interest rate $r=0$ it is first-best efficient to supply high effort and to make the corresponding high investment:

$$\text{(A0)} \qquad y(0)-1>y_0(0).$$

This ensures that high effort ($e=1$) is first-best efficient as long as the interest rate r is lower than some value $r^*(q)>0$ (i.e. $y(r)-1>y_0(r)$ for $r<r^*(q)$). To make sure that this will always be the case in the (long-run of the) first-best economy, we then have to assume that the saving rate is high enough (see Proposition 1 below).

The essential implication of first-best credit is that the allocation of productive capital between agents and therefore the equilibrium interest rate are independent from the current dispersion of wealth levels. In the absence of borrowing constraints, everybody will make the optimum investment $k(r)$ such that the current (gross) interest rate $1+r$ equals the (expected) marginal product of capital $pf'(k(r))$ (so as to maximize expected income $pf(k)-(1+r)k$), irrespective of one's initial wealth w. Rich agents will lend capital to poor agents so as to equalize the marginal product of capital throughout the economy, over all production units. Thus aggregate capital demand is $k(r)$, and since aggregate capital supply is equal to the average wealth W_t, the equilibrium interest rate at period tr_t is given by

$$k(r_t)=W_t$$

$$\text{i.e. } 1+r_t=pf'(W_t).$$

Thus, whatever the current wealth distribution $G_t(w)$, every agent will invest the average wealth W_t, so that individual (expected) income $y_{it}(w_{it})$ as a function of initial wealth w_{it} is given by

$$y_{it}(w_{it})=pf(W_t)-(1-r_t)W_t+(1+r_t)w_{it}.^7$$

and aggregate income $Y_t(G_t)$ is given by

$$Y_t(G_t)=pf(W_t).$$

7. The term $(1+r_t)w_{it}$ is individual (gross) capital income, while the $pf(W_t)-(1+r_t)W_t$, is individual labour income; the latter does not depend on one's wealth because of first-best credit; it can be regarded as the equilibrium wage rate v_t, which, as the equilibrium interest rate, depends only on the average wealth W_t.

Therefore with first-best credit, aggregate output depends only on aggregate wealth. This implies that we can track down the evolution of aggregate wealth and aggregate output without worrying about the way wealth and output are distributed: aggregate wealth at period $t+1$ W_{t+1} is given by

$$W_{t+1} = sY_t = spf(W_t). \tag{1}$$

The concavity of f together with equation (1) then implies that aggregate wealth W_t will converge to a unique long-run aggregate wealth W^*_∞, irrespective of initial aggregate wealth W_0 (and in particular irrespective of G_0(w)) (see Figure 1); W^*_∞ is given by

$$W^*_\infty = spf(W^*_\infty). \tag{2}$$

This implies that the equilibrium interest rate r_t will converge globally to a unique long-run interest rate r^*_∞ s.t. $1+r^*_\infty = pf'(W^*_\infty)$.

In order to complete the characterization of the dynamics of the wealth distribution and the interest rate with first-best credit, we must also say what long-run wealth distribution $G_\infty(w)$ will prevail, given that the long-run interest rate has to be r^*_∞. If individual income was deterministic (say, if $p=1$), all dynasties would converge to the average wealth level W^*_∞. Since we assumed idiosyncratic shocks on individual investments, there will be some positive inequality in the long-run, but this inequality will be independant of initial inequality $G_0(w)$. This is so because r^*_∞ does not depend on initial inequality, and because for any given interest rate r the wealth process follows a linear Markov process that converges globally toward a unique invariant distribution. One can see that by looking at the transitional equations:[8]

$$w_{it+1}(w_{it}) = \begin{cases} s[f(k(r)) + (1+r)(w-k(r))/p] & \text{with probability } p \\ 0 \quad \text{with probability } 1-p. \end{cases} \tag{3}$$

The concavity of the individual transition functions given by equation (3) implies that there can be no trap, i.e. that one can communicate between (any neigbourhood of) any two possible long-run wealth levels with positive probability in a finite time (see Figure 2). Thus the wealth process is globally ergodic, and the distribution $G_t(w)$ converges to the unique invariant distribution G^*_∞ associated with the interest rate r^*_∞.

We summarize these properties with the following proposition:

Proposition 1. *(A0) implies that there exists $s_0 = s_0(q)$ such that, if $s > s_0$, there exist unique levels of long-run aggregate wealth W^*_∞, aggregate output Y^*_∞, the interest rate r^*_∞ and inequality $G^*_\infty(w)$ toward which W_r, Y_t, r_t and $G_t(w)$ converge as t goes to ∞, irrespective of the initial wealth distribution $G_0(w)$.*

Note that Proposition 1 would also hold with a concave savings function $S(y)$.[9] The

8. Because of risk-neutrality, all agents are actually indifferent between all divisions of their total expected income between the lucky and the unlucky states of nature. Assuming these particular transition functions is of no consequence for the dynamics with first-best credit (the uniqueness of the long-run interest rate r^*_∞ and distribution G^*_∞ would hold with any transition function belonging to the agents' indifference curves). It does however simplify substantially the analysis of the dynamics with credit-rationing because it allows us to compute the stationary distributions (see Section 4 and the appendix).

9. See Stiglitz (1969). With convex savings however, long-run accumulation can depend on the initial distribution (see Bourguignon (1981)). We assume this away so as to better isolate the effects of credit rationing per se.

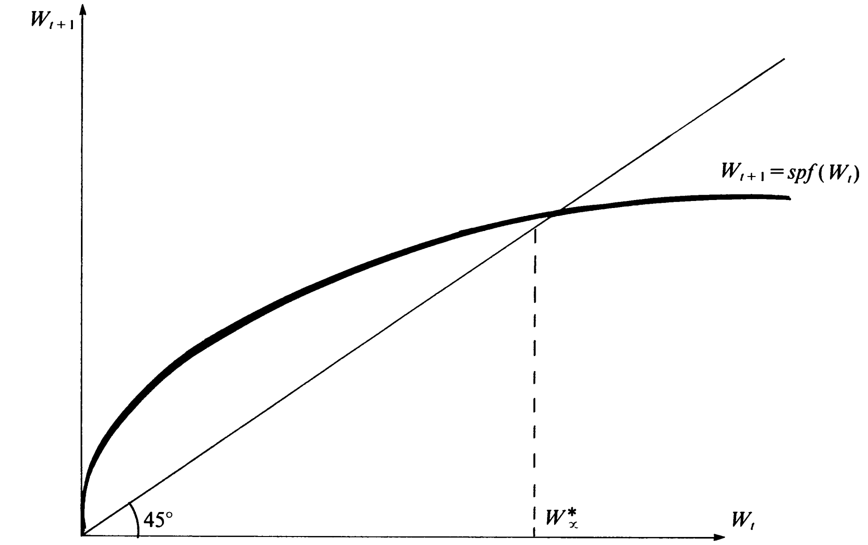

FIGURE 1
Aggregate dynamics with first-best credit

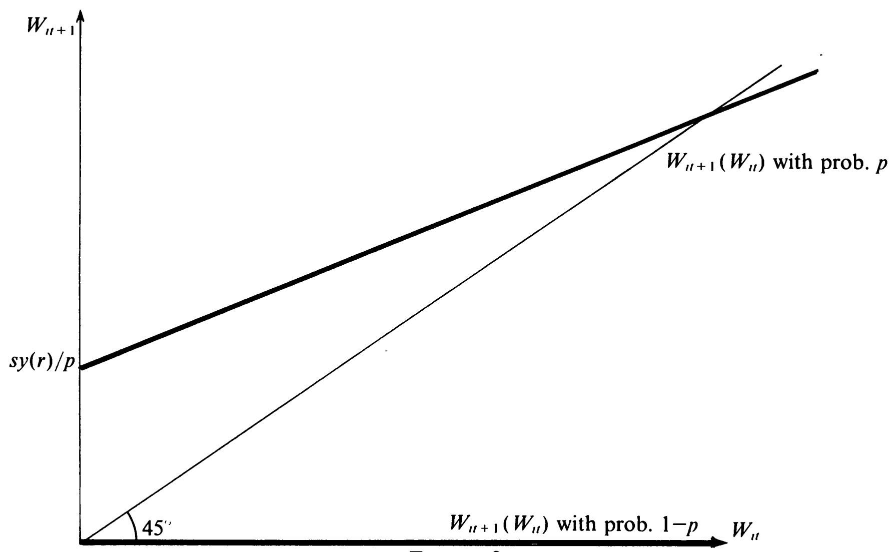

FIGURE 2
Individual transitions with first-best credit

uniqueness of the long-run interest rate, output and capital stock would also hold with the assumption that dynasties maximize an intertemporal utility function of the form $U=\sum_{s\geqq t} U(C_s)/(1+\theta)^s$.[10]

10. In that case, the unique long-run interest rate r^*_∞ would be equal to the rate of time preference θ, and the unique long-run capital stock W^*_∞ would then be given by $pf'(W^*_\infty)=\theta$, irrespective of the initial wealth distribution $G_0(w)$. Note however that in the absence of idiosyncratic shocks the long-run distribution G^*_∞ could then be any distribution such that the marginal product of the average wealth W^*_∞ is equal to θ: only aggregate variables are uniquely determined. We did not assume such dynastic preferences because they make the analysis of equilibrium credit rationing completely intractable. This is why "dynastic" models with credit constraints usually assume exogeneous non-negativity constraints for consumers and forget about credit-rationing in production (see, e.g., Aiyagari (1995)). Moreover, whether or not dynastic preferences provide a better description of actual savings behaviour is controversial (see, e.g., Solow (1994, p. 49)).

3. CREDIT-RATIONING

We now introduce credit-rationing. There are many different microeconomic theories of credit-rationing, mostly based upon the non-observability of labour input (moral hazard)[11], physical output,[12] or individual ability (adverse selection).[13] They all have the implication that one can borrow more with a higher collateral, because of the commitment value of initial wealth. Although the exact micro story does not really matter for our purposes, we choose to model credit-rationing as arising from a moral-hazard problem, following Aghion and Bolton (1997). The only point of departure from the Solow model introduced in Section 2 is that we now assume that individual labour supply ($e=0$ or 1) is no longer observable, so that lenders must check beforehand whether borrowers have adequate incentives to supply their unit of effort. We first derive the static properties of credit rationing in this model.

Assume that the current interest rate is $r>0$, and consider an agent whose initial wealth w is below the optimum investment $k(r)$ associated to r.[14] Assume also that $r>r^*(q)$, so that it is indeed first-best optimal to supply high effort ($e=1$) and to make the high investment $k(r)$.[15]

Since lenders cannot directly observe the agent's effort supply, they can provide proper incentives only by offering a financial contract specifying repayments (d_f, d_s) depending on whether the project fails (output $=0$) or succeeds (output $=f(k)$), in exchange for investing $k(r)-w$. We assume perfect competition between lenders, so that whenever a contract yielding non-negative expected profits does exist it will be offered, and only zero-profit contracts will be traded in equilibrium.

Since we assumed investment to be sunk costs, repayment d_f has to be 0 when the investment fails, while d_s will have to be whatever it takes to cover interest payments in expected terms:

$$d_f=0$$

$$d_s=(1+r)(k(r)-w)/p$$

so that

$$pd_s+(1-p)d_f=(1+r)(k(r)-w).$$

But incentives to take high effort are now distorted, and ex post (after the contract is signed) the borrower will take high effort if and only if

$$p[f(k(r))-d_s]-1>q[f(k(r))-d_s], \tag{4}$$

where the RHS is the borrower's expected net income obtained with $e=1$ minus the effort cost and the LHS is the borrower's expected net income with $e=0$. The incentive-compatibility equation (4) shows that the more the agent has to borrow (the higher $k(r)-w$), the less the agent benefits from a high probability of success, and the higher the incentive to shirk. If the incentive-compatibility condition is not satisfied (i.e. if $k(r)-w$ is too high), then lenders will anticipate that the agent will shirk and therefore will not

11. See, e.g., Aghion and Bolton (1997).
12. See, e.g., Banerjee and Newman (1993) and Galor and Zeira (1993).
13. See, e.g., Jaffee and Stiglitz (1990).
14. If $w>k(r)$, then the agent does not need any credit market to make this investment).
15. If $r>r^*(q)$ then everybody prefers to make the low investment $k_0(r)$ and to supply low effort $e=0$, and everybody can obtain sufficient credit, since borrowers cannot reduce their effort further.

invest $k(r)-w$: the agent is credit-rationed and cannot make the optimal investment $k(r)$. Rewriting equation (4), this will arise if and only if

$$(1+r)(k(r)-w)>[1+(p-q)f(k(r))]/[r(p-q)/p]$$

or

$$w<w(r)=k(r)-[pf(k(r))-p/(p-q)]/(1+r). \tag{5}$$

Note also that if $w<w(r)$ the incentive-compatibility equation (4) cannot be satisfied for any investment level k, not even if k is lower than the first-best investment $k(r)$. This is because $f(k)-r(k-w)/p$, i.e. income net of repayment in case the project succeeds, is maximal for the optimal investment $k(r)$, so that incentives to take high effort are lower for any suboptimal investment level. It follows that if an agent cannot obtain the required credit for the first-best optimal investment then the only other option is to make the low investment $k_0(r)$ and to supply minimal effort $e=0$. As was noted above, agents can always obtain sufficient credit for this low investment.

The extent to which credit rationing is binding depends however on the current interest rate r. In particular if we assume that

$$pf(k(0))-k(0)>p/(p-q),$$

which will hold if q is sufficiently small (because of assumption (A0)), then $w(r)<0$ for r sufficiently small. That is, credit rationing disappears if the interest rate is sufficiently low because the net returns become sufficiently high to give proper incentives to agents with no collateral. As r increases, $w(r)$ increases, and we prove in the appendix that for any $q>0$ there exists $r(q)<r^*(q)$ such that if the interest rate r is above $r(q)$ then $w(r)$ is positive, i.e. credit rationing becomes binding for those agents whose initial wealth is below some positive cutoff level $w(r)$. Note that q must obviously be strictly positive, otherwise there is no commitment issue and $w(r)<0$ for any $r<r^*(q)$: first-best credit obtains.

We summarize these static properties of credit rationing in the following proposition:[16]

Proposition 2. *(A0) implies that there exists $q_0>0$, $q_0<p$, such that for any q such that $0<q<q_0$, there exists $r(q)\in\,]0, r^*(q)[$ such that:*

(*i*) *If $r\leqq r(q)$, there is no credit rationing: $\forall w_i$, dynasty i can obtain sufficient credit to make the first-best optimum investment $k(r)$.*

(*ii*) *If $r(q)<r<r^*(q)$, there is some credit rationing: $\exists w(r)>0$ such that if $w_i<w(r)$, dynasty i is credit rationed and can only make the low investment $k_0(r)$; if $w_i\geqq w(r)$, dynasty i can obtain sufficient credit to make the optimum investment $k(r)$. Moreover, $w'(r)>0$ and $w(r)\to 0+$ as $r\to r(q)+$.*

(*iii*) *If $r>r^*(q)$, then everybody prefers to make the low investment $k_0(r)$ and can obtain sufficient credit (as in the first-best case).*

We can already note an important static consequence of credit-rationing with exogeneous interest rates. In the short-run higher interest rates are always bad for net borrowers and good for net lenders, both with first-best credit and credit rationing. With first-best credit however, the aggregate effect depends only the aggregate credit position: the GNP

16. In the continuous-effort version of this model (Piketty (1992)), credit rationing exhibits essentially the same properties, except that the credit rationing curve $k(w, r)$ is smoothly increasing in w instead of being discontinuous (here we have $k(w, r)=k_0(r)$ for $w<w(r)$ and $k(w, r)=k(r)$ for $w>w(r)$). This simplifies the analysis of the dynamics without changing qualitatively the results.

of an open economy that is a net lender at the current world rate $r_t \in]r(q), r^*(q)[(W_t > k(r_t))$ would rise following a positive interest rate shock $(dY_t = (W_t k(r_t))dr_t)$, and conversely. With credit-rationing, this may not be so: the GNP of the same economy (and not only its GDP) can fall since a higher rate brings the quality of credit allocation further away from the first best: $dY_t = (W_t - k(r_t))dr_t - G_t'(w(r_t))w'(r_t)[y(r_t) - y_0(r_t)]dr_t$, where the second term measures the output drop due to the increased fraction of credit-constrained agents; the aggregate effect can therefore be negative even if the country is a net lender. These are however static effects, not taking into account the dynamic effects on capital accumulation and future equilibrium interest rates, to which we now turn.

4. DYNAMICS WITH CREDIT RATIONING

We now study the dynamics of the economy with credit rationing. At each period t, given an initial distribution $G_t(w)$ the equilibrium interest rate $r_t = r(G_t)$ is given by the equality of capital demand and capital supply, where capital demand is possibly constrained by credit rationing. If aggregate wealth at period t, W_t, is sufficiently high, then the equilibrium interest rate r_t will be lower than $r(q)$, nobody will be credit-constrained in equilibrium and the equilibrium interest rate depends only on the aggregate wealth W_t:

$$pf'(W_t) < 1 + r(q), \text{ then } 1 + r(G_t) = pf'(W_t) \tag{6}$$

However, if W_t is lower so that $pf'(W_t) > 1 + r(q)$ then there has to be some credit rationing in equilibrium,[17] and the (unique) equilibrium interest rate $r_t = r_t(G_t)$ is determined by

$$\text{If } pf'(W_t) > 1 + r(q), \text{ then } r_t = r(G_t) \text{ s.t. } G_t(w(r_t))k_0(r_t) + [1 - G_t(w(r_t)]k(r_t) = W_t \tag{7}$$

In that case the equilibrium interest rate is no longer determined by "the" marginal product of capital, simply because the latter varies across production units, depending on whether they are credit-constrained or not. The entire distribution of wealth now matters, and not only aggregate wealth. This makes the dynamics of the wealth distribution and the interest rate substantially more complicated than in the no-credit-rationing case, where we could first track down the non-linear evolution of aggregate wealth (a single-dimensional state variable) without worrying about the distribution before coming back to the issue of the long-run distribution once the unique possible long-run interest rate r^*_∞ is determined (these distributional dynamics then followed a linear Markov process).

Given the equilibrium interest rate at time $t r_t = r(G_t)$ (given by equations (6) and (7)) individual transitions $w_{it+1}(w_{it})$ are the same as in the first-best world (equations (3)) for the fraction of the population which is not credit-constrained at time t (i.e. those dynasties $i \in I$ s.t. $w_{it} > w(r_r)$). The new individual transitions for those dynasties which are credit-constrained (there will be none if $r_t < r(q)$) are given by:[18]

$$\text{If } w_{it} < w(r_t),\ w_{it+1}(w_{it}) = \begin{cases} f(k_0(r)) - r(k_0(r) - w)/q & \text{with probability } q \\ 0 \quad \text{with probability } 1 - q. \end{cases} \tag{8}$$

17. If W_t is so low that $qf'(W_t) > 1 + r^*(q)$, then everybody turns to the low-effort investment $k_0(r_t)$, the equilibrium interest rate is $r_t = qf'(W_t) - 1$ and there is no credit rationing. We do not consider this case since we know from Proposition 1 that this can only be temporary under the assumption $s > s_0$.

18. As in the first-best case, risk neutrality implies that agents are indifferent between all possible ways to divide total income between the lucky and the unlucky states, and we pick these particular transition functions for simplicity (see note 8). Note that with non-verifiable this "indeterminacy" exists only for credit-constrained agents investing $k_0(r)$ and supplying low effort anyway: unconstrained borrowers making the high investment need to bear "risk" to have adequate incentives, so that the individual transitions given by equation (4) are now the only possible ones for $w_{it} > w(r)$.

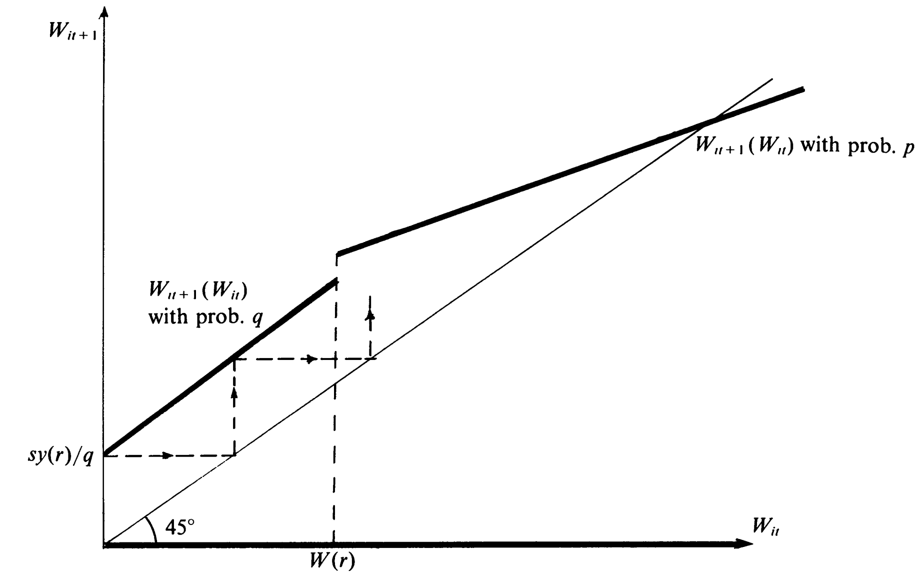

FIGURE 3
Individual transitions with credit-rationing ($r(q) < r^*$)

We represent these transition functions on Figure 3 for an equilibrium interest rate r_t for which there is some credit-rationing ($r(q) < r_t < r^*(q)$).

The equilibrium interest-rate schedule given by equations (6) and (7) and the individual transitions given in equations (3) and (8) define a non-linear aggregate transition function $G_{t+1}(G_t)$. If the economy starts with some initial distribution of wealth G_0, this defines an infinite sequence of wealth distributions and equilibrium interest rates $(G_t, r_t)_{t \geqq 0}$. We are interested in the long-run steady-states of this dynamic system, i.e. in the set of (G_∞, r_∞) such that $G_{t+1}(G_\infty) = G_\infty$ and $r_\infty = r(G_\infty)$. In the same way as in the first-best case, for any possible long-run interest rate r_∞ individual transition functions define a linear, globally ergodic Markov process converging toward a unique stationary distribution $G_{r\infty}(w)$. It follows that an interest rate r_∞ can be self-sustaining (i.e. can be a long-run steady-state interest rate) iff r is equal to the equilibrium interest rate $r(G_{r\infty})$ associated to its stationary distribution $G_{r\infty}$.

Proposition 3. *Assume* (*A*0) *and* $0 < q < q_0$. *To each possible stationary interest-rate* $r_\infty \in [0; r^*(q)]$ *corresponds a unique stationary, ergodic distribution* $G_{r\infty}(w)$. *Then* r_∞ *is a long-run steady-state interest rate of the dynamic system* $(G_{t+1}(G_t), r_t = r(G_t))$ *defined above if and only if* $r_\infty = r(G_{r\infty})$.

If the long-run interest rate r^*_∞ associated with unconstrained accumulation (equations (1) and (2)) is sufficiently low that no credit constraint ever appears (i.e. $r^*_\infty < r(q)$), then high aggregate wealth, low equilibrium interest rates and no credit rationing will be self-sustaining, so that the no-credit-rationing steady state (G^*_∞, r^*_∞) analyzed in Section 2 will also be a steady-state of the second-best economy. This will be so if the saving rate s is high enough.

However this is not in general the only possible long-run outcome of the economy. Along with this no-credit-rationing steady state and for the same parameter values there can co-exist another possible long-run steady state associated with a higher interest rate $r_\infty^{**} > r(q) > r_\infty^*$, and another stationary distribution $G_\infty^{**}(w)$ with a positive steady-state fraction $G_\infty^{**}))(w(r_\infty^{**}))$ of credit-constrained agents. For this to be true, two key conditions must hold.

First, it must be the case that the steady-state fraction of credit-constrained borrowers $G_r(w(r))$ increases sufficiently as the interest rate r goes up. In general, a higher interest rate has the effect of making both upward mobility and downward mobility less likely: it is more difficult to escape the credit-rationing interval $[0, w(r)]$ both the cutoff $w(r)$ is higher and because credit-constrained agents are net borrowers[19]), and at the same time it is more difficult to fall into this interval (because the wealthy have high interest incomes even if their investment project fails). The net effect on $G_r(w(r))$ will be positive if the first effect dominates, i.e. if the "credit-constraint effect" dominates the "interest-income effect".

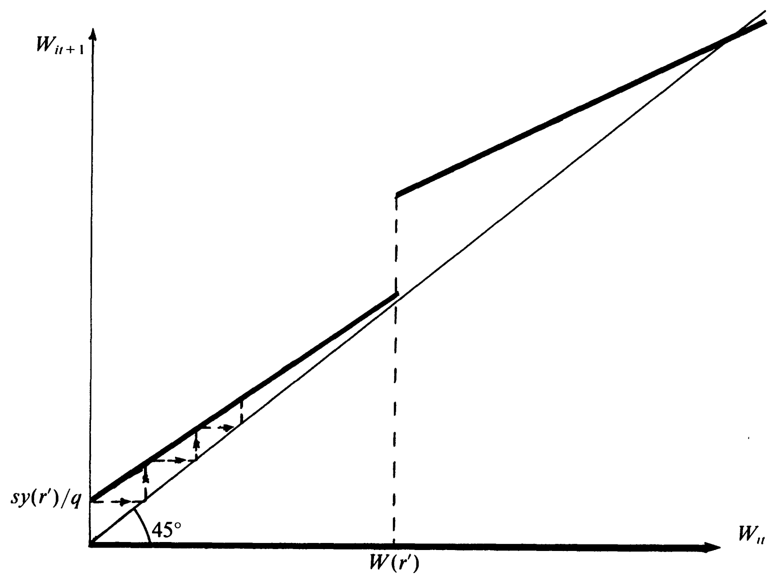

FIGURE 4

Individual transitions with credit-rationing $[r(q) < r < r' < r^*(q))]$

Because of risk neutrality and the way we modelled individual transitions, the second effect does not operate: wealthy agents always have a fixed probability p of going bankrupt, and this does not depend on the current interest rate. It follows that $G_r(w(r))$ increases with r because of the credit-constraints effect. In Figure 4 we represent the same transitions as in Figure 3 but for a higher rate $r' > r$: two consecutive successful investment periods to escape from credit rationing in Figure 3, whereas it takes much more time in Figure 4, implying a higher steady-state fraction of credit-constrained individuals. Given the way

19. We show in the appendix that $k_0(r) > w(r)$ (at least for q sufficiently small), so that credit-constrained agents are net borrowers (if $w_i < w(r)$ then $w_i < k_0(r)$). If credit-constrained agents were not net borrowers, then they could benefit from a higher interest rate and their steady-state fraction $G_r(w(r))$ may not increase with r, in which case the no-credit-rationing steady-state would be the unique steady-state under assumption (A1) (below). In any case, this property would be entirely driven by the 0–1 effort decision, and more generally it would obtain in models with non-convexities and fixed investments where credit-constrained individuals can make no investment and lend their low wealth (as in the recent literature referred to in Section 1). In contrast, credit-constrained agents are always net borrowers in the continuous-effort version of our model (Piketty (1992)): this is the natural implication of a decreasing-returns technology.

we modelled credit rationing, this effect will be particularly strong if q is small: as q tends to zero credit-constrained dynasties can make only arbitrarily small investments until they reach the threshold $w(r)$ so that the steady-state fraction of credit-constrained individuals $G_r(w(r))$ goes to 1.[20]

Next, it must be the case that the existence of a higher fraction of credit-constrainted dynasties tends to push the equilibrium interest rate up. This, together with the first condition above, will make high interest rates self-sustaining, providing that the saving rate is not too high (but higher than the minimum saving rate making low interest rates and no credit-rationing self-sustaining). Credit-constrained agents supply less capital than other agents: for a given rate r, they accumulate $sqf(k_0(r)$ while others accumulate $spf(k(r))$. But they also demand less capital: $k_0(r) < k(r)$. A higher fraction of credit-constrained agents will tend to push the equilibrium rate up if the first effect dominates, which requires the average product of capital of unconstrained agents $pf(k(r))/k(r)$ be higher than the average product of capital of credit-constrained agents $qf(k_0(r))/k_0(r)$. Given the way we modelled credit-rationing, it is sufficient to assume that:

(A1) $$k \to f(k)/kf'(k) \text{ increases with } k.$$

Under these conditions one can prove that there will exist at least one interest rate $r_\infty^{**} > r(q)$ such that r_∞^{**} and its associated stationary distribution G_∞^{**} constitute a steady-state.

Proposition 4. *Assume (A0) and (A1). Then there exists $s_1(q)$, $s_2(q)$ such that $0 < s_0(q) < s_1(q) < s_2(q)$, q_1 such that $0 < q_1 < p$, such that if $s_1(q) < s < s_2(q)$ and $0 < q < q_1$ there exists at least two steady (states r_∞^*, G_∞^*) and (r_∞^{**}, G_∞^{**}) of the dynamic system ($G_{t+1}(G_t)$, $r_t = r(G_t)$), with $r_\infty^* < r^*(q) < r_\infty^{**}$.*

One can summarize the intuition for this multiplicity in the following way. Starting from the high-accumulation, low-interest-rate steady state (r_∞^*, G_∞^*), a positive shock on the interest rate can be self-sustaining if it pushes sufficiently many agents in the credit-rationing region for a sufficiently long time, so that capital accumulation is sufficiently depressed to make high interest rates self-sustaining. The key conditions for this to happen is that credit rationing has large negative consequences for capital accumulation, which in our model is captured by a very low q.[21]

Moreover these multiple steady states can always be ranked in aggregate terms: steady states associated to higher interest rates have lower aggregate capital stock and aggregate output. This is simply because under (A1) steady-state multiplicity requires the steady-state fraction $G_r(w(r))$ of credit-constrained individuals to be increasing with r.

Proposition 5. *Assume (A0) and (A1). Then if there exist multiple steady states ($G_{\infty i}(w)$, $r_{\infty i}$) for $1 \leq i \leq n$, with $r_{\infty 1} < \cdots < r_{\infty n}$, then aggregate capital stock $W_{\infty i}$ and output levels $Y_{\infty i}$ associated to these steady-states are inversely related to the interest rate:*

$$W_{\infty 1} > \cdots > W_{\infty n} \text{ and } Y_{\infty 1} > \cdots > Y_{\infty n}$$

20. This provides the main justification for modelling credit rationing the way we did: q measures the outside option of credit-constrained people, providing us with a simple, intuitive indicator of the toughness of credit rationing.

21. Note that it is crucial that these dramatic effects of the imperfect credit market appear only for high interest rates. For example if there was no credit market at all and therefore no equilibrium interest rate, then individual transitions would not depend on the current distribution and there would be no steady-state multiplicity.

On the other hand, if the net effect of credit-constrained individuals on the interest rate is negative (i.e. if their lower capital demand $k_0(r) < k(r)$ outweighs their lower capital accumulation $spf(k_0(r)) < spf(k(r))$), which in the particular context of our model means that (A1) does not hold, then opposite phenomena could in general happen. The existence of multiple steady-state interest rates would then require the steady-state fraction of credit-constrained agents $G_r(w(r))$ to be sufficiently decreasing with r. As noted above, this could arise if credit-constrained agents are net lenders ($k_0(r) > w(r)$) or if the "interest income effect" is stronger than the "credit-constraint effect", so that higher interest rates make credit rationing a more transitory state in individual trajectories. One could then obtain high steady-state interest rates associated to high output and high wealth. We chose to focus on the opposite situation because it seems both more natural and interesting.

5. CONCLUDING COMMENTS

There is no room for long-run growth in our constant-returns accumulation model. However, our long-run level effects of short-run shocks could easily become long-run growth effects if one adds some rationale for self-sustained growth to our framework. For example if long-run growth is positively related to aggregate investment through some economy-wide externality (as in Romer(1986)), then the low-interest-rate, high-wealth-mobility steady state would exhibit faster growth than the high-interest-rate, low-wealth-mobility steady state.[22] Since stationary distributions associated to higher rates are typically more unequal (there are more credit-constrained poor and the very rich accumulate more), countries with more unequal wealth distribution would grow less, assuming that national credit markets are imperfectly integrated (so that different countries can be in different steady-states).

Our results can also contribute to the recent debate about credit cycles and the credit crunch. This literature typically treats the supply of funds as exogeneous, and shows how the (given) current availability of credit and wealth distribution determine the current allocation of capital.[23] We show how capital accumulation itself is determined by the pattern of credit allocation of the previous period, and that this interaction between credit constraints and capital accumulation can give rise to multiple equilibrium paths even with a perfectly convex, Solow-type technology. Therefore credit rationing is not only a powerful transmission mechanism (as emphasized by this literature), but can also have long-run consequences: policy shocks reducing real rates temporarily from r_∞^{**} to r_∞^{*} can be self-sustaining through the induced effects on accumulation and the distribution of new worth and credit (and conversely).

APPENDIX

Proof of Proposition 1. We note $y(r) = pf(k(r)) - (1+r)k(r)$, $y_0(r) = qf(k_0(r)) - (1+r)k_0(r)$. We have $y'(r) = -k(r)$, $y_0'(r) = -k_0(r)$; thus $y'(r) - y_0'(r) < 0$, and since $y(0) - y_0(0) > 1$ by assumption (A0) and $y(r)$, $y_0(r) \to 0$ as $r \to \infty$, there exists $r^*(q) > 0$ such that

$$y(r) - 1 > y_0(r) \quad \text{for } r < r^*(q)$$

22. To ensure that credit constraints will persist with long-run growth, one must assume that the effort cost $e = 1$ agents have to pay to get the high success probability (p) grows at the same rate as the economy. Otherwise incentive constraints would disappear in the long-run.

23. For example, Bernanke and Gertler (1989, 1993) document how credit constaints and actual investment depend on firms's net worth and how it varies across different types of firms. Theoretical models also take capital supply as given and focus upon the allocation of credit across productive units (see, e.g., Kyotaki and Moore (1993) and Holmstrom and Tirole (1993)).

and

$$y(r)-1<y_0(r) \quad \text{for } r>r^*(q).$$

Next, $dr^*_\infty/ds<0$ (by concavity of f, see Figure 2), $r^*_\infty\to\infty$ as $s\to 0$ (since $f'(\infty)=0$) and $r^*_\infty\to 0$ as $s\to\infty$ (since $f'(\infty)=0$); thus there exists $s_{01}(q)=k(r^*(q))/pf(k(r^*(q))$ such that if $s>s_{01}(q)r^*_\infty<r^*(q)$. This ensures that if $r_t<r^*(q)$ initially it will remain so, so that r^*_∞ is indeed a steady-state interest rate of the first-best economy. To ensure this is the only one, s must be high enough so that if $r_t>r^*(q)$ initially it will eventually pass above $r(q)$. In the same way as above, this will hold if $s\leq s_{02}(q)=k_0(r^*(q))/qf(k(r^*(q))$. Thus if $s\leq s_0(q)=\text{Max}\,(s_{01}(q), s_{02}(q))$, dynasties of the first-best economy always use high effort and high investment in the long-run, and W^*_∞ and s^*_∞ are the unique long-run values (we shall see below that Max $(s_{01}, s_{02})=s_{02}$ under assumption (A1)).

Finally we have to prove that $G_t(w)$ converges toward a unique $G^*_\infty(w)$ as $t\to\infty$. Since $r_t\to r^*_\infty$ and individual transition functions vary continuously with r, we just have to look at the transitions associated to r^*_∞. These transitions are linear Markovian, there exists a maximum long-run wealth level $w_m(r^*_\infty)$ as long as p is sufficiently close to 1 (see Figure 3; this follows from $s(1+r^*_\infty)=sf'(W^*_\infty)<1$; see Figure 2), they are monotonic (i.e. $w_{it+1}(w_{it})$ dominates $w_{it+1}(w'_{it})$ in the first-order stochastic sense if $w_{it}>w'_{it}$), and they verify the following "concavity property": one can find a point $w\in[0; w_m(r^*_\infty)]$ (say, $w=k(r^*_\infty)$) such that there exists $N\geqq 1$ and $\varepsilon>0$ such that proba$(w_{it+N}>w\,|\,w_{it}=0)>\varepsilon$ and proba$(w_{it+N}<w\,|\,w_{it}=w_m(r^*_\infty))>\varepsilon$. We can then apply Theorem 2 of Hopenhayn and Prescott (1992, p. 1397) to derive the existence of a unique stationary distribution $G^*_\infty(w)$ toward which $G_t(w)$ converges as $t\to\infty$, irrespective of the initial distribution $G_0(w)$.

Given the simple transitions functions of equation (3) however, we can also prove global covergence by computing directly the unique stationary distribution, including in the case where p is not sufficiently close to 1 (so that $s(1+r^*_\infty)/p>1$ although $(s(1+r^*_\infty)<1)$ and where there exists no maximum long-run wealth $w_m(r^*_\infty)$. Define $w_0=0$ and $w_{T+1}=s(f(k(r^*_\infty))+(1+r^*_\infty)(w_T-k(r^*_\infty))/p)$; this defines an infinite sequence $(w_T)_{T\geq 0}$ converging to $w_m(r^*_\infty)$ if $s(1+r^*_\infty)/p<1$ and to $+\infty$ otherwise (see Figure 2). Note $H(w)=G(w+)-G(w-)$ for any distribution $G(w)$. $\forall G_0(w)$, equation (3) implies that $G_1(0)=1-p$, and then that $H_2(w_1)=p(1-p)$, and more generally that $H_t(w_T)=p^T(1-p)$ for any $t>0$ and $T<t$. This implies that G_t converges toward G^*_∞ defined by $H^*_\infty(w_T)=p^T(1-p)\ \forall T\geqq 0$. ‖

Proof of Proposition 2. We have $w(r)=k(r)-[pf(k(r))-p/(p-q)]/(1+r)$. This gives

$$w'(r)=[pf(k(r))-p/(p-q)]/(1+r)^2$$

Moreover $[pf(k(r))-(1+r)k(r)]-[qf(k(r))-(1+r)k(r)]>y(r)-y_0(r)$, and $y(r)-y_0(r)>1$ for $r<r^*(q)$. Thus $pf(k(r))-p/(p-q)>0$ for $r<r^*(q)$, and therefore $w'(r)>0$ for $r<r^*(q)$. Define q_{01} s.t. $pf(k(0))-k(0)=p/(p-q_0)$. By assumption (A0), $q_{01}>0$. If $q>q_0$, then $w(0)<0$. Moreover $q\to w(r^*(q))$ is continuous, $w(r^*(0))=0$, and

$$dw(r^*(q))/dq=w'(r^*(q))r^{*\prime}(q)+p/[(p-q)^2(1-r^*(q)].$$

Since $r^{*\prime}(q)=-f(k_0(r^*(q)))/[k(r^*(q))-k_0(r^*(q))]$ and $k_0=0$ for $q=0$, it follows that

$$dw(r^*(q))/dq|_{q=0}=1/p(1+r^*(0))>0.$$

It follows that there exists $q_{02}>0$ s.t. for $q<q_{02}w(r^*(q))>0$. By continuity of $r\to w(r)$ it follows that for $q<q_{02}$ there exists $r(q)<r^*(q)$ s.t. $w(r(q))=0$ and $w(r)>0$ for $r\in\,]r(q); r^*(q)]$. Finally, if we take $q_0=\text{Min}(q_{01}, q_{02})$, (i) and (ii) of Proposition 2 hold for $0<q<q_0$(i.e. $r(q)>0$). ‖

Proof of Proposition 3. We apply the same direct-computation method as in Proposition 1 to prove global convergence to a unique stationary distribution $G_{r\infty}(w)$ for any given possible long-run interest rate r_∞. If $r_\infty<r(q)$, the transitions functions are the same as with first-best credit (and also if $r_\infty>r^*(q)$). If $r(q)<r_\infty<r^*(q)$, define $w_0=0$, $w_{T+1}=s(f(k_0(r_\infty))+(1-r_\infty)(k_0(r_\infty)-w_T)/q)$ if $w_T<w(r_\infty)$ and $w_{T+1}=s(f(k(r_\infty))+(1+r_\infty)(w_T-k(r_\infty))/p)$ if $w_T>w(r)$. Define $T^*>0$ s.t. $w_{T^*}<w(r_\infty)$ and $w_{T^*+1}>w(r_\infty)$. $G_{r\infty}(w)$ must be such that

$$G_{r\infty}(w(r_\infty))=(1-q)G_{r\infty}(w(r_\infty))+(1-p)(1-G_{r\infty}(w(r_\infty)))$$
$$+\cdots+q^{T^*}(1-p)(1-G_{r\infty}(w(r_\infty)))$$

that is:

$$G_{r\infty}(w(r_\infty))=(1-p)/[(1-q)/(1-q^{T^*+1})-(p-q)]$$

and

$$G_{r\infty}(0)=(1-q)G_{r\infty}(w(r_\infty))+(1-p)(1-G_{r\infty}(w(r_\infty)))$$

$$H_{r\infty}(w_T)=q^T G_{r\infty}(0) \text{ for } T\leqq T^*$$

$$H_{r\infty}(w_T)=(q^{T^*}+p^{T-T^*})G_{r\infty}(0) \text{ for } T>T^*$$

(as in the proof of Proposition 1, $H(w)=G(w+)-G(w-)$ for any distribution G).

This defines a unique stationary distribution $G_{r\infty}(w)$. In the same way as in Proposition 1, one can prove that G_t converges toward $G_{r\infty}$, $\forall G_0$.

The second part of the proposition follows directly from the definitions given above. ||

Proof of Proposition 4. In the same way as in the proof of Proposition 1, $r^*_\infty<r(q)$ if $s>s_{10}(q)=k(r(q))/pf(k(r(q)))$. Thus $r^*_\infty=r(G^*_\infty)$ if $s>s_{10}(q)$ and $q<q_0$. Moreover (A1) implies that $h:q\to k_0(r,q)/qf(k_0(r,q))$ is a decreasing function, since by definition $qf'(k_0(r,q))=r\,\forall q$, so that $h(q)=k_0 f'(k_0)/rf(k_0)$, implying that $h'(q)<0$ since $dk_0/dq>0$ and $k\to kf'/f$ decreasing by assumption. It follows that $k_0(r(q))/qf(k_0(r(q)))>k(r(q))/pf(k(r(q)))=s_{10}(q)$. Define $s_2(q)=k_0(r(q))/qf(k_0(r(q)))$, and $s_{11}(q)=k_0(r^*(q))/qf(k_0(r^*(q)))$. We just saw that $s_2(q)>s_{10}(q)$. Moreover, $s_2(q)>s_{11}(q)$ since $r^*(q)>r(q)$ and $r\to k_0(r)/qf(k_0(r))$ is a decreasing function of r. If we define $s_1(q)=\text{Max}(s_{10}(q),s_{10}(q))$, if follows that $s_1(q)<s_2(q)$, and that if $s\in\,]s_1(q);s_2(q)[$ then (i) $r^*_\infty=r(G^*_\infty)$, and (ii) $\exists r_a\in\,]r(q);r^*(q)[$ s.t. $sqf(k_0(r_q))=k_0(r_q)$.

(a) We will first prove that there exists $q_1>0$ s.t. if $0<q<q_1$ there exists $r\in\,]r(q);r^*(q)[$ s.t.

$$S(r)<0$$

where $S(r)$ is the "excess supply function" (see below) defined by

$$S(r)=S(q,r)=[1-G_r(w(r))][spf(k(r))-k(r)]+G_r(w(r))[sqf(k_0(r))-k_0(r)]$$

($G_r(w)$ is the stationary distribution associated to interest rate r; we know that it is unique from Proposition 3.)

Pick up some sequence $r[q]\in\,]r(q);r_q[$ for q small such that $r_q-r[q]$ converges monotonically toward 0.

First note that $q\to G_r(w(r))$ tends to 1 as $q\to 0+$ for any $r\in\,]r(q);r(q);r_q[$. This is because

$$G_r(w(r))\geqq(1-q)G_r(w(r))+(1-p)(1-G_r(w(r))$$

that is:

$$\mathrm{G_r(w(r))>(1-p)/(1-p+q)>1-q/(1-p)}.$$

Next note that $spf(k(r))-k(r)<M=spf(k(r^*(0)))-k(r^*(0))\;\forall r\leqq r^*(0)$.

Define $Z(q,r)=sqf(k_0(r,q))-k_0(r,q)$. Since $r\in\,]r(q);r_q[$, $Z(q)<0\forall q$. Moreover $r[q]\text{-}r_q\to 0$ as $q\to 0$ (since $r(q)$, $r^*(q)\to r^*(0)$ as $q\to 0$), and (ii) above then implies that $Z(q,r[q])\to 0-$ as $q\to 0+$.

We now have: $S(q,r[q])<[q/(1-p)]M+(1-q/(1-p)]Z(q,r[q])$. Both terms tend to 0 to $q\to 0$ but they are opposite sign: intuitively, the negative second term will will if $r[q]$ tends sufficiently slowly to r_q as $q\to 0$.

To prove that this will always be so, pick a s.t. $0<a<(1-p)/M$. Assume first that $Z(q,r[q])<-q/a$. Then:

$$S(q,r[q])<q(M/(1-p)-1/a)+[a/(1-p)]q^2 \quad \text{for } q \text{ small.}$$

Since $M/(1-p)-1/a<0$ by a assumption, $S(q,r[q])<0$ for q small. That is, there exists $q'_1>0$ such that if $q<q'_1$ and $Z(q,r[q])<-q/a$, then $S(q,r[q])<0$.

Assume now that $Z(q,r[q])>-q/a$, i.e. $-aZ(q,r[q])<q$. Since $q\to Z(q,r)$ increases with q, it follows that $Z(-aZ(q,r[q]),r[q])<Z(q,r[q])$. This implies that

$$S(-aZ(q,r[q]),<[-aZ(q,r[q])/(1-p)]M+(1+Z(q,r[q])/(1-p))Z(r[q] \quad \text{for } r[q])q \text{ small.}$$

Since $1-aM/(1-p)>0$, if follows that $S(-aZ(q,r[q]),r[q])<0$ for q small. That is, there exists $q''_1>0$ such that if $q<q''_1$ and $Z(q,r[q])>-q/a$, then $S(-aZ(q,r[q]),r[q])<0$. Finally, let $q_1=\min(q'_1,-aZ(q''_1,r^*[q''_1])$. Then if $q<q_1$, there exists $r\in\,]r(q);r(q)[$ s.t. $S(r)<0$ (take $r=r[q]$ in case $Z(q,r[q])<-q/a$, and take $r=r[q']$ with q' s.t. $-aZ(q',r[q'])=q$ in case $Z(q,r[q])>-\mathrm{q/a}$). Thus we have proved the desired property.

(b) Now consider the mapping $r\to S(r)$. If $r\to G_r(w(r))$ was continuous, then this mapping would be continuous, and since $S(r[q])<0$ and $S(r_q)>0$ this would prove the existence of $r^{**}_\infty\in\,]r[q];r_q[$ such that $S(r^{**}_\infty)=0$. This would establish that $r^{**}_\infty=r(G^{**}_\infty)$, since $S(r)$ is the long-run excess supply for capital:

$$S(r) = W_r - [(1 - G_r(w(r))k(r) + G_r(w(r))k_0(r)]$$

with

$$W_r = (1 - G_r(w(r)))spf(k(r)) + G_r(w(r))spf(k_0(r)).$$

However $r \rightarrow G_r(w(r))$ is discontinuous. Fortunately one can show that it is lower semi-continuous, thereby establishing it must cross the horizontal axis. To prove that consider the number of lucky periods $T^*(r)$ it takes to pass over $w(r)$ as a function of the candidate long-run interest rate r (T^* is defined in the proof of Proposition 3). For $T \leqq T^*(r)$ we have $w_{T+1} = s(f(k_0(r)) - (1-r)(k_0(r) - w_T)/q)$, i.e. $w_{T+1} + w^*(r) = [s(1+r)/q](w_T + w^*(r)]$, with $w^*(r) = [sf(k_0(r)) - s(1+r)k_0(r)/q]/[s(1+r)/q - 1]$. For q small enough $w^*(r) > 0$, and $w^{*\prime}(r) < 0$. Since $w_T = w^*(r)[(s(1+r)/q)^T - 1]$, it follows that

$$T^*(r) = \text{Int}\,[\log(1 + \text{w(r)}/\text{w}^*(\text{r}))/\log(\text{s}(1+\text{r})/\text{q}].$$

To ensure that $T^*(r)$ is increasing with r, it is sufficient to make sure that credit-constrained individuals are net borrowers: if $w(r) < k_0(r)$ then $dw_T/dr < 0$ for $T \leqq T^*$, so that $T^*(r)$ increases with r (since the threshold $w(r)$ increases as r increases; see Proposition 2). This, in turn, will be the case in particular if q is small enough: for $r \in]r(q); r_q[$, both $k_0(r)$ and $w(r)$ tend to 0 as $q \rightarrow 0$; but $dk_0/dq = -f'(k_0)/qf''(k_0) \rightarrow +\infty$ as $q \rightarrow 0$, whereas $dw(r)/dq \rightarrow 1/p$ as $q \rightarrow 0$ (see the proof of Proposition 2); it follows that $k_0(r) > w(r)$, at least for q small enough.

Thus for q small enough $T^*(r)$ is an increasing and therefore upper semi-continuous function of r, and so is $G_r(w(r))$ (see the proof of Proposition 3 for $G_r(w(r))$ as a function of T^*). Since $r \rightarrow S(r)$ is increasing for a fixed fraction $G_r(w(r))$ of credit-constrained agents but decreases as $G_r(w(r))$ "jumps" to a higher value, it follows that $r \rightarrow S(r)$ is lower semi-continuous. Thus for $q < q_1$, $\exists r_\infty^{**} \in]r[q]; r_q[$s.t. $S(r_\infty^{**}) = 0$. ||

Proof of Proposition 5. Step (b) of the proof of Proposition 4 established that $G_r(w(r))$ was increasing with r for r in the credit-rationing interval, i.e. $r \in]r(q); r^*(q)[$, for q small enough. This is also true for $r < r(q)$ and $r > r^*(q)$ (for $r < r(q)$ there is no credit rationing so $G_r(w(r)) = 0$, and for $r > r^*(q)$ everybody invests $k_0(r)$ so $G_r(w(r)) = 1$). If (r, G_r) constitute a steady state of $(G_{t+1}(G_t), r_t = r(G_t))$, then steady-state aggregate output Y_r and wealth W_r are given by

$$Y_r = G_r(w(r))qf(k_0(r)) + (1 - G_r(w(r)))pf(k(r))$$

$$W_r = sY_r = G_r(w(r))k_0(r) + (1 - G_r(w(r)))k(r).$$

Since $k_0(r) < k(r)$ and both decrease as r increases, it follows that Y_r and W_r are decreasing functions of r. Moreover assumption (A1) alone implies that $G_r(w(r))$ must be increasing for multiple steady-states to exist: we saw in the proof of Proposition 4 that the "long-run excess supply function $S(r)$ increases with r if $G_r(w(r))$ is kept fixed and that it decreases if $G_r(w(r))$ rises; so even outside the case analysed in Proposition 4 (i.e. q small enough), steady-state multiplicity implies "steady-state aggregate ranking" if (A1) holds. ||

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