Wealth Distribution in Finite Life with Investment Risk

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April 14, 2009
Why does the wealth distribution in U.S. displays the following characteristics?

- A Gini coefficient as high as 0.78
- Skewness to the right
- Pareto tail

I propose a parsimonious model including bequest motives and investment risk to match these three features in the data.
Three Features

- Gini and Quintiles (Castaneda, Diaz-Gimenez and Rios-Rull (2003))

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Top tail

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- Pareto tail. Using the richest sample of the U.S., the Forbes 400, during 1988-2003 Klass et al. (2006) find that the top end of the wealth distribution obeys a Pareto law with an average exponent of 1.49.
A continuous time OLG model

- There is a continuum of agents in the economy.
- Finite life. Each agent gives birth to one child when he dies.
  - Scenario (i) No uncertainty of life span. Thus the age cohort has equal size.
  - Scenario (ii) Age-dependent death rate. Thus the economy has realistic age cohort distribution.
- Agents may have bequest motives.
- Investment risk within lifetime. This work is different from Behabib and Bisin (2008).
Agent’s problem

\[
\max_{c(t), \phi(t)} \left\{ E_t \int_t^T \frac{c(s)^{1-\gamma}}{1-\gamma} e^{-\theta(s-t)} ds + \chi \frac{[(1 - \zeta) w(T)]^{1-\gamma}}{1-\gamma} e^{-\theta(T-t)} \right\}
\]

s.t. \[ dw(t) = [(1 - \tau) r w(t) + (1 - \tau) \rho - (1 - \tau) r \phi(t) w(t) - c(t) + \omega + \Gamma] dt + (1 - \tau) \sigma \phi(t) w(t) dz(t) \]

where \( \omega \) is the wage rate, \( \Gamma \) is the government lump-sum transfer. \( \tau \) is capital income tax rate. \( \zeta \) is estate tax rate.

The agent’s human wealth

\[
h(t) = \int_t^T (\omega + \Gamma) e^{-(1-\tau)r(s-t)} ds
\]
The agent’s policy functions are

\[ c(t) = a(t)^{-\frac{1}{\gamma}}[w(t) + h(t)] \]

\[ \phi(t)w(t) = \frac{(1 - \tau)\rho - (1 - \tau)r}{\gamma\sigma^2(1 - \tau)^2}(w(t) + h(t)) \]

where

\[ a(t) = \left( \begin{aligned} (\chi(1 - \zeta)^{1-\gamma}) & \frac{1}{\gamma} \\ + & \left( (\chi(1 - \zeta)^{1-\gamma}) \frac{1}{\gamma} + \frac{1}{1-\frac{1}{\gamma}[(1-\tau)r + \frac{1}{2}\frac{(\rho - r)^2}{\gamma\sigma^2}] - \frac{\theta}{\gamma} } \right) \\ \exp \left\{ \frac{1-\gamma}{\gamma} \left[ (1 - \tau)(T - t) + \frac{1}{2}\frac{(\rho - r)^2}{\gamma\sigma^2} \right] - \frac{\theta}{\gamma} \right\} (T - t) - 1 \end{aligned} \right)^\gamma \]

And

\[ d(w(t) + h(t)) = \left( (1 - \tau)r + \frac{(\rho - r)^2}{\gamma\sigma^2} - a(t)^{-\frac{1}{\gamma}} \right)(w(t) + h(t))dt + \frac{\rho - r}{\gamma\sigma}(w(t) + h(t))dz(t) \]
Wealth accumulation within lifetime

- Let \( x(t) \) be the total wealth, i.e. the sum of physical wealth and human wealth.

\[
x(t) = w(t) + h(t)
\]

From proposition 1, we know

\[
dx(t) = \left( (1 - \tau)r + \frac{(\rho - r)^2}{\gamma \sigma^2} - a(t)^{-\frac{1}{\gamma}} \right) x(t) dt + \frac{\rho - r}{\gamma \sigma} x(t) dz(t)
\]

- The end-of-life wealth is

\[
w(T) = x(T)
\]

\[
= \left( \chi (1 - \zeta)^{1-\gamma} \right)^{\frac{1}{\gamma}} a(0)^{-\frac{1}{\gamma}} \exp\left[ \left( \frac{(1 - \tau)r - \theta}{\gamma} + \frac{(\rho - r)^2}{2\gamma \sigma^2} \right) T \right]
\]

\[
+ \frac{\rho - r}{\gamma \sigma} z(T) \right] x(0)
\]
Intergenerational wealth connection

- Let $T, 2T, 3T, \cdots, nT, \cdots$ be the born time of generation 1, 2, 3, $\cdots$, $n, \cdots$. Let

$$x_1 = x(T), x_2 = x(2T), x_3 = x(3T), \cdots, x_n = x(nT), \cdots$$

- Bequest Movement

Illustration of Bequest Movement in a Linage
Equation describing wealth connection

- Agent’s starting wealth includes received bequest and human wealth

\[
x_{n+1} = (1 - \zeta)w((n + 1)T) + h(0)
\]

\[
= \left( \frac{\chi(1 - \zeta)}{a(0)} \right)^{\frac{1}{\gamma}} \exp\left[ \left( \frac{(1 - \tau)r - \theta}{\gamma} + \frac{(\rho - r)^2}{2\gamma\sigma^2} \right) T + \frac{\rho - r}{\gamma\sigma} z(T) \right] x_n
\]

+ h(0)

- Let

\[
\rho_{n+1} = \left( \frac{\chi(1 - \zeta)}{a(0)} \right)^{\frac{1}{\gamma}} \exp\left[ \left( \frac{(1 - \tau)r - \theta}{\gamma} + \frac{(\rho - r)^2}{2\gamma\sigma^2} \right) T + \frac{\rho - r}{\gamma\sigma} z(T) \right]
\]

Note that \( \rho_{n+1} \) is lognormally distributed.

Thus

\[
x_{n+1} = \rho_{n+1} x_n + h(0)
\]
By Sornette (2006) and Goldie (1991), the starting wealth displays an asymptotic Pareto upper tail, i.e.

\[ P(x(0) > x) \sim x^{-\mu} \]

where

\[ \mu = \gamma \left( \frac{1}{T} \log \left( \frac{a(0)}{\chi(1-\zeta)} \right) + \theta - (1 - \tau) r \left( \frac{\rho - r}{2\sigma^2} \right) - 1 \right) \]
The Pareto tail of the starting wealth distribution implies that wealth distribution conditional on any age also displays Pareto tail with the same exponent.

Hump shape of wealth accumulation.

If

\[ 0 < \frac{\gamma - 1}{\gamma} \left( (1 - \tau)r + \frac{(\rho - r)^2}{2\gamma \sigma^2} \right) + \frac{\theta}{\gamma} < \frac{1}{(\chi(1 - \zeta)^{1-\gamma})^{\frac{1}{\gamma}}} \]

then the mean wealth of the age cohort has a hump shape.
The wealth distribution of the whole economy displays a Pareto tail of the same exponent as that of the starting wealth distribution.
Age-dependent death rate

- Let $\pi(t), t \in [0, T]$ be the death rate of agent. Define

$$G(t) = \int_t^T \pi(s) \, ds$$

and

$$\pi(v, t) = \frac{\pi(v)}{G(t)}$$

- Agent’s problem

$$\max_{c, P, \phi} \left\{ E_t \int_t^T \pi(v, t) \left[ \int_t^v \frac{c(s)^{1-\gamma}}{1-\gamma} e^{-\theta(s-t)} \, ds \right. \right.$$

$$\left. + \chi \frac{[(1-\zeta)Z(v)]^{1-\gamma}}{1-\gamma} e^{-\theta(v-t)} \right] \, dv \right\}$$

s.t. $dw(t) = [(1-\tau)rw(t) + ((1-\tau)\alpha - (1-\tau)r)\phi(t)w(t) - c(t) - P(t) + \omega + \Gamma] \, dt$

$$+ (1-\tau)\sigma\phi(t)w(t) \, dz(t)$$
Intergenerational connection

Now let \( t_1, t_2, t_3, \ldots, t_n, \ldots \) be the born time of generation 1, 2, 3, \ldots, \( n, \ldots \). Let

\[
x_1 = x(t_1), \ x_2 = x(t_2), \ x_3 = x(t_3), \ldots, \ x_n = x(t_n), \ldots
\]

We have

\[
x_{n+1} = \rho_{n+1} x_n + h(0)
\]

where

\[
\rho_{n+1} = \frac{(\chi(1 - \zeta))^{\frac{1}{\gamma}}}{a(0)^{\frac{1}{\gamma}}} \exp\left\{ \left[ \frac{(1 - \tau)r - \theta}{\gamma} + \frac{(\alpha - r)^2}{2\gamma\sigma^2} \right](t_{n+1} - t_n) + \frac{\alpha - r}{\gamma\sigma} (z(t_{n+1}) - z(t_n)) \right\}
\]

- Pareto tail
Calibrated Economy

- Parameters. $\theta = 0.04, \ r = 0.01, \ \gamma = 2.5, \ \alpha = 0.08, \ \sigma = 0.2, \ \chi = 15, \ \zeta = 0.19, \ \tau = 0.25, \ t \in [20, 91]$.

- Gini and Lorenz curve.
  - Gini and Quintiles

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<td>2.6</td>
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Pareto Tail

- Pareto exponent. $\mu = 1.6545$.
- Log-Log plot
Wealth Accumulation Path

![Wealth Accumulation Path Graph](image-url)
Density of Wealth Distribution (model)
Plan to Do

- Tax effect
- Disentangle inequality
- Wealth dispersion and consumption dispersion with aging