

Introduction to Economic History

(Master PPD & APE)

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Course Notes on Basic Theoretical Models of Economic Growth, Wealth Accumulation and Concentration

(check [on line](#) for updated versions)

Roadmap of course notes

- The standard growth model: output convergence, not income or wealth convergence
- How can we explain aggregate accumulation, i.e. the equilibrium level of $\beta = W/Y$?
- The pure lifecycle model
- The dynastic model
- The random-shocks model: explaining the relation between equilibrium wealth concentration & $r-g$
- Explaining the long-run evolution of inheritance & share of inherited wealth

For more details on these theoretical models, see Piketty-Zucman, « Wealth & Inheritance in the Long Run », [Handbook of Income Distribution 2015](#)

The standard growth model: output convergence, not income & wealth cv

- The standard (neoclassical) growth model has many limitations: one-good model (no relative asset price), perfectly competitive markets with full information, little attention to inequality, etc.
- But it is a useful starting point to think about growth mechanics
- Output $Y = F(K,L)$ = production function

with K = capital input (= non-human capital: land, buildings, equipment, robots, patents, etc.)

and L = labor input (= human capital: efficient labor units = active population x labor productivity)

- Exemple: Cobb-Douglas production function: $F(K,L)=K^\alpha L^{1-\alpha}$
→ one needs capital K and labor L to grow at the same rate in order to have balanced long-run growth of Y
(also true with more general production functions: see lecture 3)

- Basic logic of the convergence result in the standard growth model: if capital can freely flow from rich to poor countries, and if labor productivity is the same everywhere, then per capita output will be the same everywhere = « convergence »
- This result requires strong assumptions: perfect competition, one-good model, no specialization effect (core/periphery models), no colonial extraction, etc.
- But even if these strong assumptions are all satisfied, the point is that the standard growth model predicts output cv , not income or wealth cv : one can end up with permanent wealth inequality, within countries as well as between countries: foreign-owned countries, political instability and redistribution cycles (Africa, South America)
- Asian miracles were induced by domestic saving, diffusion of knowledge and education, pro-development policies and public investment, not by capital flowing from rich to poor countries

Basic references on the standard growth model

- R. Solow, “A Contribution to the Theory of Economic Growth’, [QJE 1956](#); S. Kuznets, “Modern Economic Growth: findings and reflections”, [AER 1973](#)
- R. Barro, X. Sala-i-Martin, *Economic Growth*, MIT Press, 2004, [Chap.1-2](#)
- R. Gordon, « Is US economic growth over? Faltering innovation confronts the six headwinds », [NBER 2012](#); “The Demise of US Economic Growth: Restatement, Rebuttal, and Reflections”, [NBER 2014](#) ; « Secular Stagnation: the Supply-side View », [AER 2015](#)
- L. Summers, « Demand-Side Secular Stagnation», [AER 2015](#)
- B. Eichengreen, « Secular Stagnation: the Long View », [AER 2015](#)
- C. Jones, P. Romer, « The new Kaldor facts : ideas, institutions, population and human capital », [AEJ 2010](#) ; P. Krugman, A. Venables, « Globalization and the Inequality of Nations », [QJE 1995](#)

- More on standard growth models: see Solow [QJE 1956](#), Barro-Sala-i-Martin [2004 Chap.1-2](#), Jones-Romer [AEJ 2010](#)
- Most important steady-state formula to remember: modified Golden rule formula $r = \theta + \gamma g$ ($r > g$)
 (θ = rate of time preference, γ = curvature of utility function)
- $r > g$ does not mean ever-rising inequality: it simply means that ownership makes sense, i.e. that capital owners only need to reinvest a fraction g/r (<1) of their capital income to ensure that their wealth rises as fast as the size of the economy
- I.e. if $r=5\%$, $g=1\%$, capital owners only need to reinvest 20% of their capital income and can consume 80%
- $r < g$ would imply that they need to reinvest more than 100% of capital income (« dynamic inefficiency »: too much capital)
- Here I only provide simple numerical examples

- Two countries A and B with same population & labor productivity $L_A=L_B$
- **Exemple 1: A and B have same per capita wealth $W_A=W_B=200\ 000\text{€}$**
- No need for capital flows between countries A and B: each country has the same per capita domestic capital $K_A=K_B=200\ 000\text{€}$
- Assume $Y=F(K,L)$ is such that per capita output $Y_A=Y_B=25\ 000\text{€}$,
i.e. capital-output ratio $\beta = K/Y = \text{wealth-output ratio } W/Y = 800\%$
- No net foreign wealth: per capita income = per capital output = $25\ 000\text{€}$
- Assume growth rate $g = 1\%$ (population + productivity growth) and rate of return to capital $r = 5\%$ (marginal product of capital + preferences)
- Capital share $\alpha = Y_K/Y = rK/Y = r \times \beta = 40\%$: in both count., labor income $Y_{LA}=Y_{LB}=15\ 000\text{€}$, capital income $Y_{KA}=Y_{KB}=10\ 000\text{€}$ ($=5\% \times 200\ 000\text{€}$)
- Balanced (steady-state) growth: K must rise at same speed as Y and L
- If $g=1\%$, $r=5\%$, one needs to reinvest a fraction $g/r=20\%$ of Y_K ($2\ 000\text{€}$),
and one can consume a fraction $1-g/r=80\%$ ($8\ 000\text{€}$)
 $\rightarrow Y = 25\ 000\text{€} = S + C = 2\ 000\text{€} (8\%) + 23\ 000\text{€} (92\%)$
- With $g=2\%$, $r=5\%$, one needs to reinvest a fraction $g/r=40\%$ of Y_K , etc.

- **With full equality (within & between countries), the fact that $r > g$ is not a pb at all:** it simply means that everybody needs to save and reinvest a fraction g/r of Y_K so that K rises at the same speed as Y and L (steady-state growth), & can consume a fraction $1-g/r$ of capital income = this is the purpose of K accumulation and ownership: one can consume more than without K accumulation
- $r < g$ would be a pb: one would need to reinvest more than Y_K in order to keep K rising at same speed as Y and L , which makes no sense: « dynamic inefficiency », i.e. over-accumulation of K
($r < g$ impossible in infinite-horizon models; possible in OLG models)
- But with inequality between individuals (shocks to rates of return, labor incomes, demographics, etc.), a higher gap between r & g tends to amplify shocks and wealth concentration (see lecture 3)
- **What about impact of $r - g$ on inequality between countries?**

- **Exemple 2 (unequal countries): $W_A=400\ 000\text{€}$, $W_B=0\text{€}$**
- With free capital flows, half of country A's wealth is invested in country B, so that each country still has the same per capita domestic capital $K_A=K_B=200\ 000\text{€}$ and the same per capita **output $Y_A=Y_B=25\ 000\text{€}$**
- The difference is that now country B's capital is owned by country A: **income Y_B^* in country B = labor income $Y_{LB} = 15\ 000\text{€}$,** while **income Y_A^* in country A = $Y_{LA} + Y_{KA} + Y_{KB} = 35\ 000\text{€}$**
- Balanced growth: country B doesn't save (& consumes 15 000€), while country A saves a fraction g/r of $Y_{KA} + Y_{KB}$ (& consumes the rest)
- If $g=1\%$, $r=5\%$, $Y_A^* = 35\ 000\text{€} = S + C = 4\ 000\text{€} (8\%) + 31\ 000\text{€} (92\%)$

- **Market forces can lead to output convergence (under certain conditions), but not to convergence of wealth, income & welfare:** in standard models, any initial level of wealth inequality is self-sustaining
- Higher gap between g & r implies higher steady-state inequality of consumption and welfare (if $g \approx r$, then all Y_K needs to be reinvested)
 - **Only solution: country B needs to save more** (not easy since country B is poorer than country A → more natural to accumulate debt)
...or to expropriate country A! (→ large foreign assets often come with political and military domination, so as to avoid expropriation: colonies)

- **Exemple 3 (v. unequal countries): $W_A=600\ 000\text{€}$, $W_B=-200\ 000\text{€}$ (debt)**
 - With free capital flows, half of country A's wealth is again invested in country B, so that each country still has same domestic capital $K_A=K_B=200\ 000\text{€}$ and the same per capita **output $Y_A=Y_B=25\ 000\text{€}$**
 - The difference is that now country B's capital is owned by country A, and that in addition count. B needs to repay interest payments of its foreign debt ($r \times D_B = 10\ 000\text{€}$ if $r=5\%$ and $D_B=200\ 000\text{€}$)
income Y_B^* in country B = labor income $Y_{LB} - rD_B = 5\ 000\text{€}$,
while income Y_A^* in country A = $Y_{LA} + Y_{KA} + Y_{KB} + rD_B = 45\ 000\text{€}$
 - Balanced growth: count. B doesn't save (& consumes a frac. g/r of rD_B), while count. A saves fraction g/r of $Y_{KA}+Y_{KB}+rD_B$ (& consumes the rest)
 - If $g=1\%$, $r=5\%$, $Y_B^* = 5\ 000\text{€} = S + C = -2\ 000\text{€} + 7\ 000\text{€}$,
while $Y_A^* = 45\ 000\text{€} = S + C = 6\ 000\text{€} + 39\ 000\text{€}$
- **There's nothing in standard economic models that prevents extreme inequality to persist forever, especially if $g \ll r$:** possibility of permanent inequality between countries (or dynasties), with some countries (or dynasties) working for ever for others >> difficult to justify and regulate

How can we explain $\beta = W/Y$?

- We first need a theory of why people own wealth: if economic agents only care about current consumption, then they should not own any wealth, i.e. $\beta=0$. So we need a dynamic model (at least two periods) where agents care about the future.
- OLG model: agents maximize $U(c_t, c_{t+1})$, where c_t = young-age consumption (working age) & c_{t+1} = old-age consumption
- Depending on utility function $U(.,.)$ (rate of time preference, etc.), demographic parameters, etc., one obtains different saving rates and long run β (see « Modigliani triangle » formula)
- Pb with the pure life-cycle model: individuals are supposed to die with zero wealth; in the real world, inherited wealth is also important
- Models with utility for bequest: $U(c_t, w_{t+1})$, where c_t = lifetime consumption (young + old) & w_{t+1} = bequest (wealth) left to next generation
- Depending on the strength of the bequest motive in utility function $U(c,w)$, one can obtain any saving rate and long run β

Harrod-Domar-Solow formula: $\beta = s/g$

- Exemple: if agents maximize $U(c_t, \Delta w_t = w_{t+1} - w_t)$, then with $U(c, \Delta) = c^{1-s} \Delta^s$, we get a fixed saving rate $s_t = s$, and $\beta_t \rightarrow \beta = s/g$
(i.e. $\text{Max } U(c_t, \Delta w_t)$ under $c_t + \Delta w_t \leq y_t \rightarrow \Delta w_t = s y_t$)
- **More generally, this is what we get in any one-good capital accumulation model, whatever the saving motives and utility functions behind the saving rate s_t :**
- I.e. assume that: $W_{t+1} = W_t + s_t Y_t$
 \rightarrow dividing both sides by Y_{t+1} , we get: $\beta_{t+1} = \beta_t (1 + g_{wt}) / (1 + g_t)$
With $1 + g_{wt} = 1 + s_t / \beta_t =$ saving-induced wealth growth rate
 $1 + g_t = Y_{t+1} / Y_t =$ total income growth rate (productivity + population)
- If saving rate $s_t \rightarrow s$ and growth rate $g_t \rightarrow g$, then:
 $\beta_t \rightarrow \beta = s/g$
- Exemple: if $s = 10\%$, $g = 2\%$, $\beta_t \rightarrow \beta = 500\%$
- This is a pure accounting identity: $\beta = 500\%$ is the only wealth-income ratio such that a saving rate of 10% of income corresponds to a growth rate of 2% of the capital stock
- **Intuition: the more you save, the more you accumulate, especially in a slow-growth economy**
(on these models, see [Piketty-Zucman 2015](#) & [Piketty-Zucman 2014 section 3](#))

Another special case: the dynastic model

- Pure dynastic model = model with infinite horizon and fixed rate of time preference = individuals behave as if they were infinitely lived
- Discrete time version: $U_t = \sum_{t \geq 0} U(c_t)/(1+\theta)^t$ (θ = rate of time preference)
- Budget constraint: $\sum_{t \geq 0} c_t/(1+r)^t \leq \sum_{t \geq 0} y_t/(1+r)^t$
- r_t = rate of return = $f'(k_t)$ = borrowing interest rate (perfect capital markets)
- Closed economy, representative agent: ind. wealth w_t = per capita capital stock k_t
- First-order condition: $U'(c_{t+1})/U'(c_t) = (1+\theta)/(1+r)$
- Assume $U(c) = c^{1-\gamma}/(1-\gamma)$, i.e. $U'(c) = c^{-\gamma}$, ($U(c) = \log(c)$ if $\gamma=1$)
- FO condition: $c_{t+1} = c_t [(1+r)/(1+\theta)]^{1/\gamma}$
- Intuition: if $r > \theta$, then agents want to postpone consumption to the future (conversely if $r < \theta$), and all the more so if γ close to 0, i.e. $U(c)$ close to linear
- γ = curvature of $U(c)$ (risk aversion coefficient), $1/\gamma$ = intertemporal elasticity of substitution
- Steady-state growth path: $y_t = y_0 (1+g)^t$, $k_t = k_0 (1+g)^t$, $c_t = c_0 (1+g)^t$
 - $1+r = (1+\theta) \times (1+g)^\gamma$ (with continuous time: $r = \theta + \gamma g$)
 - if $g=0$, then $r=\theta$ ($>g$): rate of return is entirely determined by preferences

- With Cobb-Douglas production function $y=f(k)=k^\alpha$, then $r=f'(k)=\alpha k^{\alpha-1}$, so that capital income $rk=\alpha y$, i.e. capital-income ratio $\beta=k/y= \alpha/r$
- I.e. if $r = \theta + \gamma g$, then $\beta=\alpha/(\theta + \gamma g)$
- Exemple: if $g=0$, $\theta=5\%$ and $\alpha=25\%$, then $\beta= \alpha/\theta =500\%$
- I.e. if lower θ (more patient), higher β
- In effect, in the dynastic model, agents save a fraction g/r of their capital income rk , so that their wealth rises at rate g , like the rest of the economy (i.e. with $g=1\%$ and $r=5\%$, they save $1/5=20\%$ of their capital income, and eat the rest

→ saving rate $s = \alpha g/r$

capital-income ratio $\beta = s/g= \alpha/r =g=0$

= special case of Harrod-Domar-Solow formula

- **To summarize: Harrod-Domar-Solow formula $\beta = s/g$ is a pure accounting formula and is valid with any saving motive and utility function**
- **Wealth increase in the utility function:** $\text{Max } U(c_t, \Delta w_t = w_{t+1} - w_t)$
 → if $U(c, \Delta) = c^{1-s} \Delta^s$, then fixed saving rate $s_t = s$, $\beta_t \rightarrow \beta = s/g$
 (i.e. $\text{Max } U(c_t, \Delta w_t)$ under $c_t + \Delta w_t \leq y_t \rightarrow \Delta w_t = s y_t$)
- **Total wealth or bequest in the utility function:** $\text{Max } U(c_t, w_{t+1})$
 → if $U(c, w) = c^{1-s} w^s$, then $w_{t+1} = s(w_t + y_t)$, $\beta_t \rightarrow \beta = s/(g+1-s) = s'/g$
 (with $s' = s(1+\beta) - \beta =$ corresponding saving rate out of income)
- **Pure OLG lifecycle model:** saving rate s determined by demographic structure (more time in retirement \rightarrow higher s), then $\beta_t \rightarrow \beta = s/g$
- **Dynastic utility:**
 $\text{Max } \sum U(c_t)/(1+\theta)^t$, with $U(c) = c^{1-1/\gamma}/(1-1/\gamma)$
 → unique long rate rate of return $r_t \rightarrow r = \theta + \gamma g > g$
 → long run saving rate $s_t \rightarrow s = \alpha g/r$, $\beta_t \rightarrow \beta = \alpha/r = s/g$

(on these models, see [Piketty-Zucman 2014 section 3](#))

The pure lifecycle model

- Modigliani triangle formula: useful formula to compute the quantity of pension wealth that one needs to accumulate for old-age purposes (as a function of demographic and economic parameters)
- Pure life-cycle model = individuals die with zero wealth (no inheritance), wealth accumulation is entirely driven by life-cycle motives (i.e. savings for retirement)
- Simplest model to make the point: fully stationary model $n=g=r=0$ (zero population growth, zero economic growth, zero interest rate = capital is a pure storage technology and has no productive use)
(see Modigliani, “Life Cycle, Individual Thrift and the Wealth of Nations”, [AER 1986](#))
- Age profile of labor income:
- Note Y_{La} = labor income at age a
- $Y_{La} = Y_L > 0$ for all $A < a < A+N$ (A = adulthood; $A+N$ = retirement age)
- $Y_{La} = 0$ for all $A+N < a < A+D$ ($A+D$ = age at death)

- I.e. people become adult and start working at age A , work during N years, retire at age $A+N$, and die at age $A+D$: labor length = N , retirement length = $D-N$
(say: $A=20$, $A+N=60$, $A+D=70$, i.e. $N=40$, $D-N=10$: $N/D = 40/50 = 80\%$, i.e. they spend 80% of their adult life working and 20% in retirement)
- Per capita (adult) national income $Y = NY_L/D$
- Preferences: full consumption smoothing
(say, $U = \sum_{A < a < A+D} U(C_a)/(1+\theta)^a$, with $\theta=0$)
- Everybody fully smoothes consumption to $C=NY_L/D$ (= per capita output Y)
- In order to achieve this they save during labor time and dissave during retirement time
- Note $S_a = \text{savings} (= Y_{La} - C)$
- We have:

$$S_a = (1-N/D)Y_L > 0 \quad \text{for all } A \leq a \leq A+N$$

$$S_a = -NY_L/D < 0 \quad \text{for all } A+N \leq a \leq A+D$$
- I.e. agents save during their working life; during retirement, they consume their past saving and die with zero wealth

- Note W_a = wealth at age a
- We get the following wealth accumulation equation:

$$W_a = (a - A)(1 - N/D)Y_L \quad \text{for } A < a < A + N$$

$$W_a = N(1 - N/D)Y_L - (a - A - N)NY_L/D \quad \text{for } A + N < a < A + D$$

>>> “hump-shaped” (inverted-U) age-wealth profile,
 Maximum wealth at age $a = A + N$, with $W_a = N(1 - N/D)Y_L$
 Then W_a back to 0 for $a = A + D$

- Average wealth is given by (simple triangle area formula):
 $W = N/D \times N(1 - N/D)Y_L/2 + (D - N)/D \times N(1 - N/D)Y_L/2$
- I.e. average $W = (D - N)Y/2$

- **Proposition: Aggregate wealth/income ratio $W/Y = (D-N)/2 =$ half of retirement length = “Modigliani triangle formula”**
 - E.g. if retirement length $D-N = 10$ years, then $W/Y = 500\%$
(and if $D-N = 20$ years, then $W/Y=1000\%$...)
 - Lessons from Modigliani triangle formula:
 - (1) pure life-cycle motives (no bequest) can generate large and reasonable wealth/income ratios
 - (2) aggregate wealth/income ratio is independent of income level and solely depends on demographics (previous authors had to introduce relative income concerns in order to avoid higher savings and accumulation in richer economies)
- Pb = one never observes so much lifecycle wealth (pension funds = at most 100-150% Y) (public pensions, inconsistencies..)

Note that in this stationary model, aggregate savings = 0: i.e. at every point in time positive savings of workers are exactly offset by negative savings of retirees; but this is simply a trivial consequence of stationnarity: with constant capital stock, no room for positive steady-state savings

- **Extension to population growth $n > 0$** : then the savings rate s is > 0 : this is because younger cohorts (who save) are more numerous than the older cohorts (who dissave)
 - Check: with population growth at rate $n > 0$, proportion of workers in the adult population = $(1 - \exp(-nN)) / (1 - \exp(-nD))$ ($> N/D$ for $n > 0$)
 - I.e. $s(n) = 1 - (N/D) / [(1 - \exp(-nN)) / (1 - \exp(-nD))] > 0$
 - Put numbers: in practice this generates savings rates that are not so small, e.g. for $n = 1\%$ this gives $s = 4,5\%$ for $D - N = 10$ yrs retirement, and $s = 8,8\%$ for $D - N = 20$ yrs retirement (keeping $N = 40$ yrs)
 - Wealth accumulation: $W/Y = s/n = s(n)/n$, i.e. wealth/income ratio = savings rate/population growth
- >> for $n = 0$, $W/Y = (D - N)/2$; for $n > 0$, $W/Y < (D - N)/2$; i.e. W/Y rises with retirement length $D - N$, but declines with population growth n

- Put numbers into the formula: $W/Y=452\%$ instead of 500% for $N=40, D-N=10, n=1\%$.
- Intuition: with larger young cohorts (who have wealth close to zero), aggregate wealth accumulation is smaller; mathematically, $s(n)$ rises with n , but less than proportionally; of course things would be reversed if N was small as compared to D , i.e. if young cohorts were reaching their accumulation peak very quickly
- Also, this result depends upon the structure of population growth: aging-based population growth generates a positive relationship between population growth and wealth/income ratio, unlike in the case of generational population growth

- **Extension to economic growth $g > 0$** : then $s > 0$ for the same reasons as the population growth: young cohorts are not more numerous, but they are richer (they have higher lifetime labor income), so they save more than the old dissave
- **Extension to positive capital return $r > 0$** : other things equal, the young need to save less for their old days (thanks to the capital income $Y_K = rW$; i.e. now $Y = Y_L + Y_K$); if $n = g = 0$ but $r > 0$, then one can easily see that aggregate consumption C is higher than aggregate labor income Y_L , i.e. aggregate savings are smaller than aggregate capital income, i.e. $S < Y_K$, i.e. savings rate $s = S/Y < \text{capital share } \alpha = Y_K/Y$

($s < \alpha$ = typically what we observe in practice, at least in countries with $n + g$ small)

- **Main limitations of the lifecycle model: it generates too much pension wealth and too little wealth inequality.** I.e. in the lifecycle model, wealth distribution is simply the mirror image of income distribution - while in practice wealth distribution is always a lot more unequal than income distribution.
- Obvious culprit: the existence of inherited wealth and of multiplicative, cumulated effects induced by wealth transmission over time and across generations. This can naturally generate much higher wealth concentration.

The dynastic model

- Pure dynastic model = individuals maximize dynastic utility functions, as if they were infinitely lived; death is irrelevant in their wealth trajectory, so that they die with positive wealth, unlike in the lifecycle model:
- Dynastic utility function: $U_t = \sum_{t \geq 0} U(c_t)/(1+\theta)^t$ ($U'(c) > 0$, $U''(c) < 0$)
- Infinite-horizon, discrete-time economy with a continuum $[0;1]$ of dynasties.
- For simplicity, assume a two-point distribution of wealth.
- Dynasties can be of one of two types: either they own a large capital stock k_t^A , or they own a low capital stock k_t^B (with $k_t^A > k_t^B$).
- The proportion of high-wealth dynasties is equal to λ (and the proportion of low-wealth dynasties is equal to $1-\lambda$), so that the average capital stock in the economy k_t is given by:

$$k_t = \lambda k_t^A + (1-\lambda)k_t^B$$

- Output is given by a standard production function $Y_t = F(K_t, L_t)$
- Output per labor unit is given by $y_t = Y_t / L_t = f(k_t)$ ($f'(k) > 0$, $f''(k) < 0$), where $k_t = K_t / L_t =$ average capital stock per capita of the economy at period t . Markets for labor and capital are assumed to be fully competitive, so that the interest rate r_t and wage rate v_t are always equal to the marginal products of capital and labor:

$$r_t = f'(k_t) \text{ and } v_t = f(k_t) - r_t k_t$$

- **Proposition:** (1) In long-run steady-state, the rate of return r and the average capital stock k are uniquely determined by the utility function and the technology (irrespective of initial conditions): in steady-state, r is necessarily equal to θ , and k must be such that : $f'(k) = r = \theta$
- (2) Any distribution of wealth (k^A, k^B) such as average wealth = k is a steady-state
- The result comes directly from the first-order condition:
- $U'(c_t) / U'(c_{t+1}) = (1+r_t) / (1+\theta)$
- I.e. if the interest rate r_t is above the rate of time preference θ , then agents choose to accumulate capital and to postpone their consumption indefinitely ($c_t < c_{t+1} < c_{t+2} < \dots$) and this cannot be a steady-state.
- Conversely, if the interest rate r_t is below the rate of time preference θ , agents choose to desaccumulate capital (i.e. to borrow) indefinitely and to consume more today ($c_t > c_{t+1} > c_{t+2} > \dots$). This cannot be a steady-state either.

- Note: if $f(k) = k^\alpha$ (Cobb-Douglas), then long run $\beta = k/y = \alpha/r$
 - Note: in steady-state, $s=0$ (zero growth, zero savings)
 - Average income: $y = v + rk = f(k) =$ average consumption
 - High-wealth dynasties: income $y^A = v + rk^A$ (=consumption)
 - Low-wealth dynasties: income $y^B = v + rk^B$ (=consumption)
- >>> everybody works the same, but some dynasties are permanently richer and consume more
- Dynastic model = completely different picture of wealth accumulation than lifecycle model
 - **In the pure dynastic model, wealth accumulation = pure class war**
 - **In the pure lifecycle model, wealth accumulation = pure age war**
 - In pure dynastic model, any wealth inequality is self-sustaining (including slavery: assume huge dynastic debt $k^B = -v/r$) (Graeber)
 - In pure lifecycle model, zero wealth inequality (if zero labor inequality)

- Dynastic model with positive (exogenous) productivity growth g : $Y_t = F(K_t, H_t)$ with $H_t = (1+g)^t L_t$ = human capital
- Modified Golden rule: $r = \theta + \sigma g$

With: $1/\sigma$ = IES (intertemporal elasticity of substitution) = constant coefficient if $U(c) = c^{1-\sigma}/(1-\sigma)$

(σ = curvature of $U(\cdot)$ = risk aversion coefficient)

- Typically $\sigma > 1$, so $r = \theta + \sigma g > g$
- But even if $\sigma > 1$, r has to be $> g$ in the dynastic, infinite-horizon model: otherwise transversality condition is violated (present value of future incomes = infinite)
- In steady-state, i.e. dynasties save a fraction g/r of their capital income (and consume the rest), so that their capital stock grows at rate g , i.e. at the same rate as labor productivity and output
- Aggregate saving rate $s = \alpha g/r < \alpha$
- Aggregate wealth-income ratio $\beta = s/g = \alpha/r$

- Where does the modified Golden rule come from?
- This comes directly from the first-order condition:
- $U'(c_t)/U'(c_{t+1}) = (1+r_t)/(1+\theta)$
- With $U(c) = c^{1-\sigma}/(1-\sigma)$, $U'(c) = c^{-\sigma} \rightarrow c_{t+1}/c_t = [(1+r_t)/(1+\theta)]^{1/\sigma}$
- Intuition: the desired consumption growth rate rises with r_t (high r_t = it is worth postponing consumption; low r_t = it is worth consuming more now), and all the more so if the IES is high
- In steady-state, consumption must grow at the same rate as the size of the economy: as $t \rightarrow \infty$, $c_{t+1}/c_t \rightarrow 1+g$
- Therefore $1+r_t \rightarrow 1+r = (1+\theta)(1+g)^\sigma$

With θ, g small (or in continuous time models), this is equivalent to :

$$r \approx \theta + \sigma g$$

In effect, if $r > \theta + \sigma g$, then agents want their consumption to rise faster than g (\rightarrow too much k accumulation, so that $r_t \downarrow$); while if $r < \theta + \sigma g$, then agents want their consumption to rise less fast than g (\rightarrow too much borrowing, so that $r_t \uparrow$)

- **Is it surprising that $r > g$ for ever?**
- No: this is obvious with $g=0 : r = \theta > 0$
- This is also what standard economic models predict with positive growth $g>0$
- And this is what we always observe in human history (at least in the absence of capital shocks, wars, taxation, etc., see below)
- Typically, during 19c: $g = 1\%$, $r = 5\%$, $s = 8\%$,
 $\beta = s/g = 800\%$, $\alpha = r \beta = 40\%$
- Wealth holders simply need to save a fraction $g/r = 20\%$ of their capital income so that their wealth rises by 1% per year, i.e. as fast as national income
- This is what wealth is here for: if you need to reinvest all your capital income in order to preserve your relative wealth position, then what's the point of holding wealth?

The random-shocks model

- Pb with the dynastic model = any level of inequality can be self-sustaining (zero mobility)
(other pb = unclear whether $r = \theta$ in the real world; in fact, it is unclear whether real-world agents really have a θ : dynamic inconsistencies, hyperbolic discounting, see e.g. [Giglio et al 2013](#))
- In the real world, there is always positive wealth mobility, because there are all sorts of random shocks: demographic (number of children, age at death, etc.), rates of return, bequest tastes, labor productivities, etc.
- Models with « ergodic » shocks = there's always a positive probability to move from any two wealth levels w_{ti} and w_{t+1i}
- Consequently there always exists a unique long-run, steady-state distribution of wealth $\phi(w)$; but for given shocks the inequality of this steady-state distribution is an increasing function of $r - g$
(where r = net-of-tax rate of return, g = growth rate)

- Finite-horizon, bequest-in-the-utility-function model = middle ground with pure lifecycle model and infinite-horizon dynastic model = more flexible and suitable model to study wealth dynamics
- Simplified version of the wealth-in-the-utility, finite-horizon model, with random shocks on saving tastes:
 - each generation lives exactly one period
 - each individual i in generation t receives labor income y_{Lt} + capitalized inherited wealth $(1+r)w_{ti}$, and maximizes $V_{ti}(c,w)=(1-s_{ti})\log(c)+s_{ti}\log(w)$ (or equivalently $V_i(c,w)=c^{s_{ti}}w^{1-s_{ti}}$) in order to allocate his total resources between consumption c_{ti} and end-of-life wealth w_{t+1i}

→ Individual-level transition equation for wealth: $w_{t+1i} = s_{ti} [y_{Lt} + (1+r) w_{ti}]$
 with: s_{ti} = saving taste = randomly drawn from a distribution with mean $s = E(s_{ti})$

- Aggregate transition equation: $w_{t+1} = s [y_{Lt} + (1+r) w_t] = s [y_t + w_t]$
 with $y_t = f(k_t) + r (w_t - k_t)$ = national income $\approx (1+g)^t y_0$ in the long run
 → $\beta_t = w_t/y_t \rightarrow \beta = s/(g+1-s) = s'/g$ (with $s' = s(1+\beta) - \beta$)

Q.: Where does the wealth distribution $\phi(w)$ converge to?

A.: $\phi \rightarrow$ Pareto distribution with exponent \uparrow as $\text{var}(s_{ti})$ and $r-g \uparrow$

- Define $z_{ti} = w_{ti}/w_t =$ normalized wealth and $\omega = s(1+r)/(1+g) < 1$
- The transition equation can be rewritten:

$$z_{t+1i} = s_{ti}/s [(1-\omega) + \omega z_{ti}]$$

- Assume binomial random tastes:
 - $s_{ti} = s^* > 0$ with proba $p > 0$ ("wealth-lovers")
 - $s_{ti} = 0$ with proba $1-p$ ("consumption-lovers")
 - $s = E(s_{ti}) = ps^*$
 - If $s_{ti} = 0$, then $z_{t+1i} = 0 \rightarrow$ children with consumption-loving parents receive no bequest
 - If $s_{ti} = s^*$, then $z_{t+1i} = s^*/s [(1-\omega) + \omega z_{ti}] \rightarrow$ children with wealth-loving parents receive positive bequests growing at rate ω/p across generations
- \rightarrow after many successive generations with wealth-loving parents (or more generally with high demographic or returns shocks), inherited wealth can be very large
- Non-explosive aggregate path: $\omega < 1$
 - Non-explosive aggregate path with unbounded distribution of normalized inheritance: $\omega < 1 < \omega^* = \omega/p$

- Therefore the steady-state distribution $\phi(z)$ looks as follows:
- $z=z_0=0$ with proba $1-p$ (children with zero-wealth-taste parents)
- $z=z_1=(1-\omega)/p$ with proba $(1-p)p$ (children with wealth-loving parents but zero-wealth-taste grand-parents)
- ...
- $z=z_{k+1}=(1-\omega)/p + (\omega/p)z_k > z_k$ with proba $(1-p)p^{k+1}$ (children with wealth-loving ancestors during the past $k+1$ generations)
- We have:
- $z_k = [(1-\omega)/(\omega-p)] [(\omega/p)^k - 1] \approx [(1-\omega)/(\omega-p)] (\omega/p)^k$ as $k \rightarrow +\infty$
- $1-\Phi(z_k) = \text{proba}(z \geq z_k) = \sum_{k' > k} (1-p)p^{k'} = p^k$
- That is, as $z \rightarrow +\infty$, $\log[1-\Phi(z)] \approx a(\log[z_0] - \log[z])$, i.e. $1-\Phi(z) \approx (z_0/z)^a$
- With Pareto coefficient $a = \log[1/p]/\log[\omega/p] > 1$
and inverted Pareto coefficient $b = a/(1-a) > 1$

- For given p :
- As $\omega \rightarrow 1$, $a = \log[1/p]/\log[\omega/p] \rightarrow 1$ and $b \rightarrow +\infty$ (infinite inequality)
- I.e. an increase in $\omega = s(1+r)/(1+g)$ means a larger wealth reproduction rate ω^* for wealth-lovers, i.e. a stronger amplification of inequality
(conversely, as $\omega \rightarrow p$, $a \rightarrow +\infty$ and $b \rightarrow 1$ (zero inequality))
- For given ω : as $p \rightarrow 0$, $a \rightarrow 1$ and $b \rightarrow +\infty$ (infinite inequality) (a vanishingly small fraction of the population gets an infinitely large shock)
(conversely, as $p \rightarrow \omega$, $a \rightarrow +\infty$ and $b \rightarrow 1$ (zero inequality))
- **Proposition: The inequality of inheritance is an increasing function of $r-g$**
- **Note.** What matters in the formula is the net-of-tax rate of return: bequest taxes - and more generally capital taxes - reduce the rate of wealth reproduction, and therefore reduce steady-state wealth concentration.

Key finding: small changes in $r - g$ can have huge impact on long-run inequality

(see Piketty-Zucman "[Wealth and Inheritance in the Long-Run](#)", 2014, section 5.4 for mathematical details and simple empirical calibration)

Note 3. The same ideas and formulas for Pareto coefficients work for different kinds of shocks.

Primogeniture: shock = rank at birth; the richest individuals are the first born sons of first born sons of first born sons etc.; see [Stiglitz *Econometrica* 1969](#)

Family size: shock = number of siblings; the richest individuals are the single children of single children of single children etc.; see [Cowell 1998](#) (more complicated formula)

Rates of return: shock = r_{t_i} instead of s_{t_i} = exactly the same multiplicative wealth process as with taste shocks → Pareto upper tails in the limit, see e.g. [Benhabib-Bisin-Zhu 2011, 2013, Nirei 2009](#)

Note 4. With primogeniture (binomial shock), the formula is exactly the same.

See e.g. Atkinson-Harrison 1978 p.213 (referred to in [Atkinson-Piketty-Saez 2011](#) p.58), who generalize the [Stiglitz 1969](#) formula and get: $a = \log(1+n)/\log(1+sr)$

This is the same formula as $a = \log[1/p]/\log[\omega^*]$: $1+n$ = population growth, so probability that a good shock occurs - i.e. being the eldest son = $1/(1+n) = p$; $1+sr$ = net-of-tax reproduction rate in case a good shock occurs = ω^* .

Note 5. The [Cowell 1998](#) result is more complicated because families with many children do not return to zero (unless infinite number of children), so there is no closed form formula for the Pareto coefficient a , which must solve the following equation: $\sum (p_k k/2) (2\omega/k)^a = 1$, where p_k = fraction of parents who have k children, with $k=1,2,3,\text{etc.}$, and ω = average generational rate of wealth reproduction.

Note 6. More generally, one can show that for any random multiplicative process $z_{t+1i} = \omega_{ti} z_{ti} + \varepsilon_{ti}$, where μ_{ti} = i.i.d. multiplicative shock with mean $\omega = E(\omega_{ti}) < 1$, ε_{ti} = additive shock (possibly random), then the steady-state distribution has a Pareto upper tail with coefficient a , which must solve the following equation: $E(\omega_{ti}^a) = 1$ (see [Nirei 2009, p.9](#) and [Nirei-Aoki 2014](#))

Special case: $p (\omega/p)^a = 1$, i.e. $a = \log(1/p) / \log(\omega/p)$.

More generally, as long as $\omega_{ti} > 1$ with some positive probability, there exists a unique $a > 1$ s.t. $E(\omega_{ti}^a) = 1$.

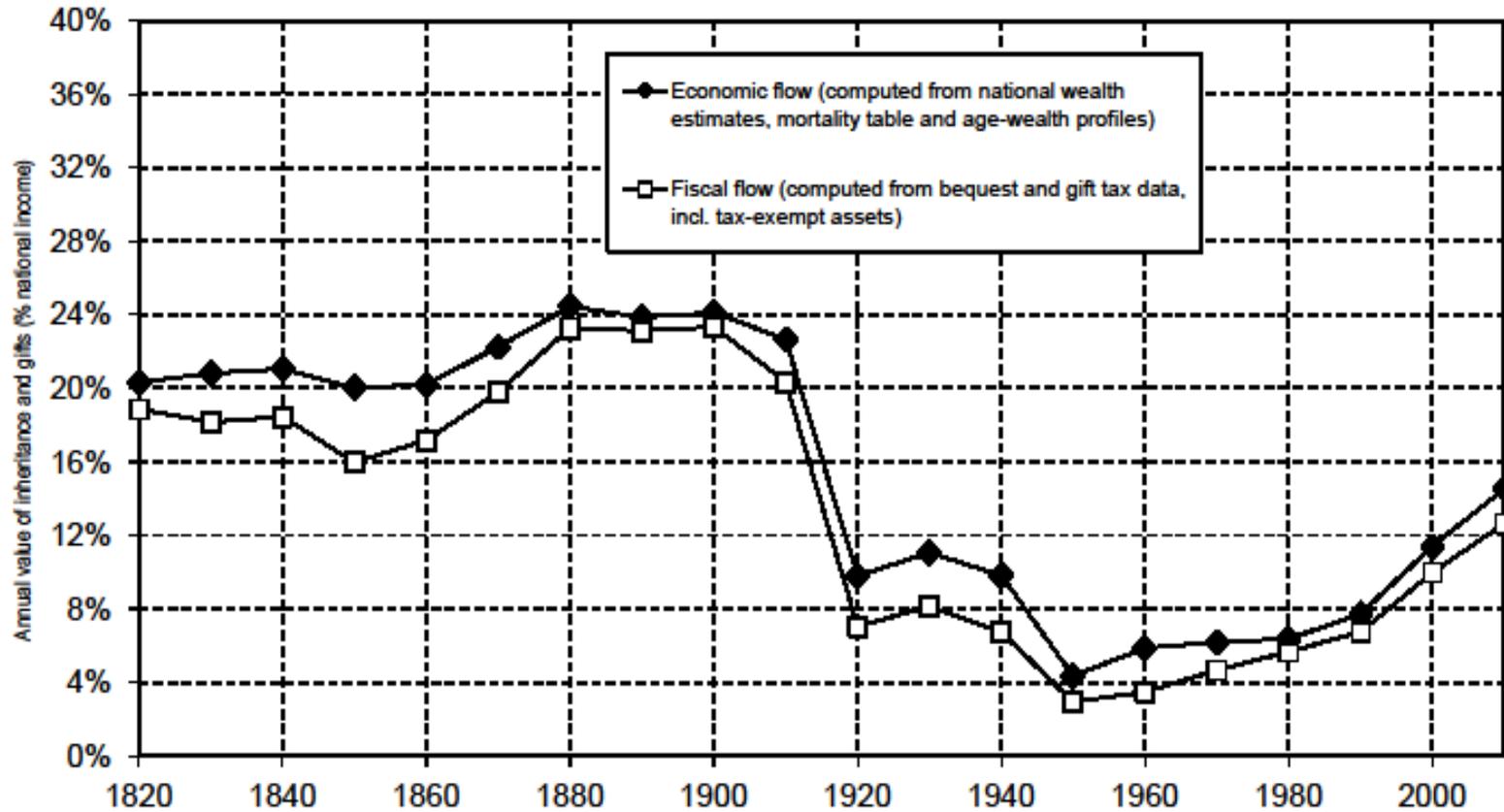
One can easily see that for a given average $\omega = E(\omega_{ti}) < 1$, $a \rightarrow 1$ if the variance of shocks goes to infinity (and $a \rightarrow \infty$ if the variance goes to zero).

Explaining the long run evolution of inheritance

- The multiplicative r-g process is already powerful with fine horizon; but it is even more powerful with inheritance; otherwise the process starts over again at each generation (wealth inequality can still be very unequal if large multiplicative shocks and long life span)
- What evidence do we have? French data is particularly good and allows to compute the inheritance flow in two independent ways: fiscal flow vs economic flow (for full details, see « On the long-run evolution of inheritance: France 1820-2050 » [QJE 2011](#))
- The economic flow of inheritance can be computed by using the following formula: $b_y = \mu m \beta$

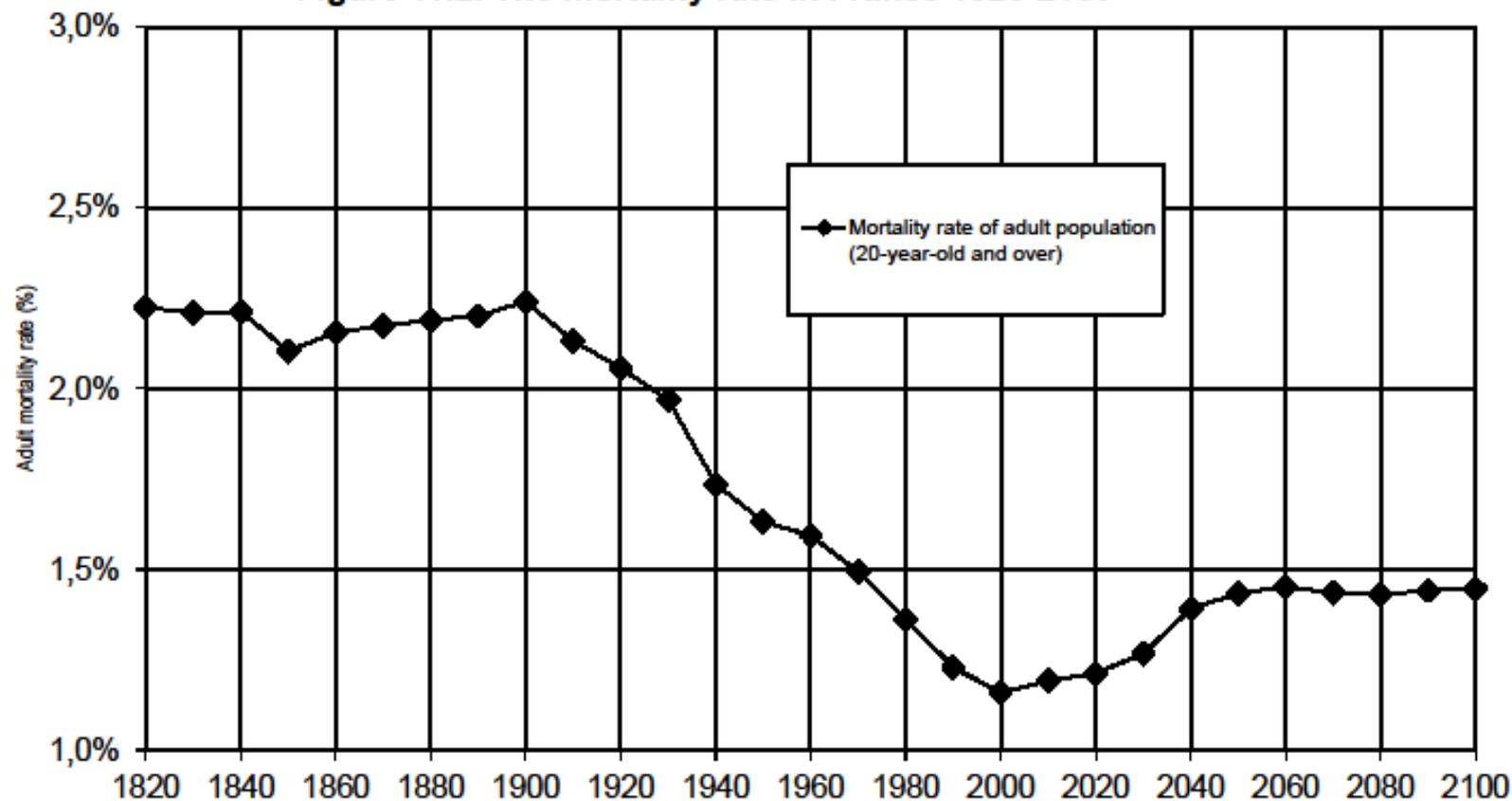
- The economic flow of inheritance can be computed by using the following formula: $b_y = \mu m \beta$
with $b_y = B/Y$, B = aggregate annual flow of inheritance (bequest + inter vivos gifts), Y = national income
 $\beta = W/Y$ = aggregate wealth-income ratio
 m = mortality rate (if people never die, there's no inheritance..)
 μ = ratio between average wealth at death and average wealth of the living (if pure lifecycle mode, people die with no wealth: $\mu=0$)
- If $\beta = 600\%$, $m=1,5\%$, $\mu=100\%$, then $b_y = 9\%$
- If $\beta = 600\%$, $m=1,5\%$, $\mu=200\%$, then $b_y = 18\%$
- We have analyzed the evolution of β in lectures 2-4
- The long run evolution of m is pretty clear: as life expectancy goes from 60 to 80 year-old, adult mortality rates go from $m=1/40=2,5\%$ to $m=1/60=1,7\%$ (for constant population)
- What about the evolution of μ ?

Figure 11.1. The annual inheritance flow as a fraction of national income, France 1820-2010



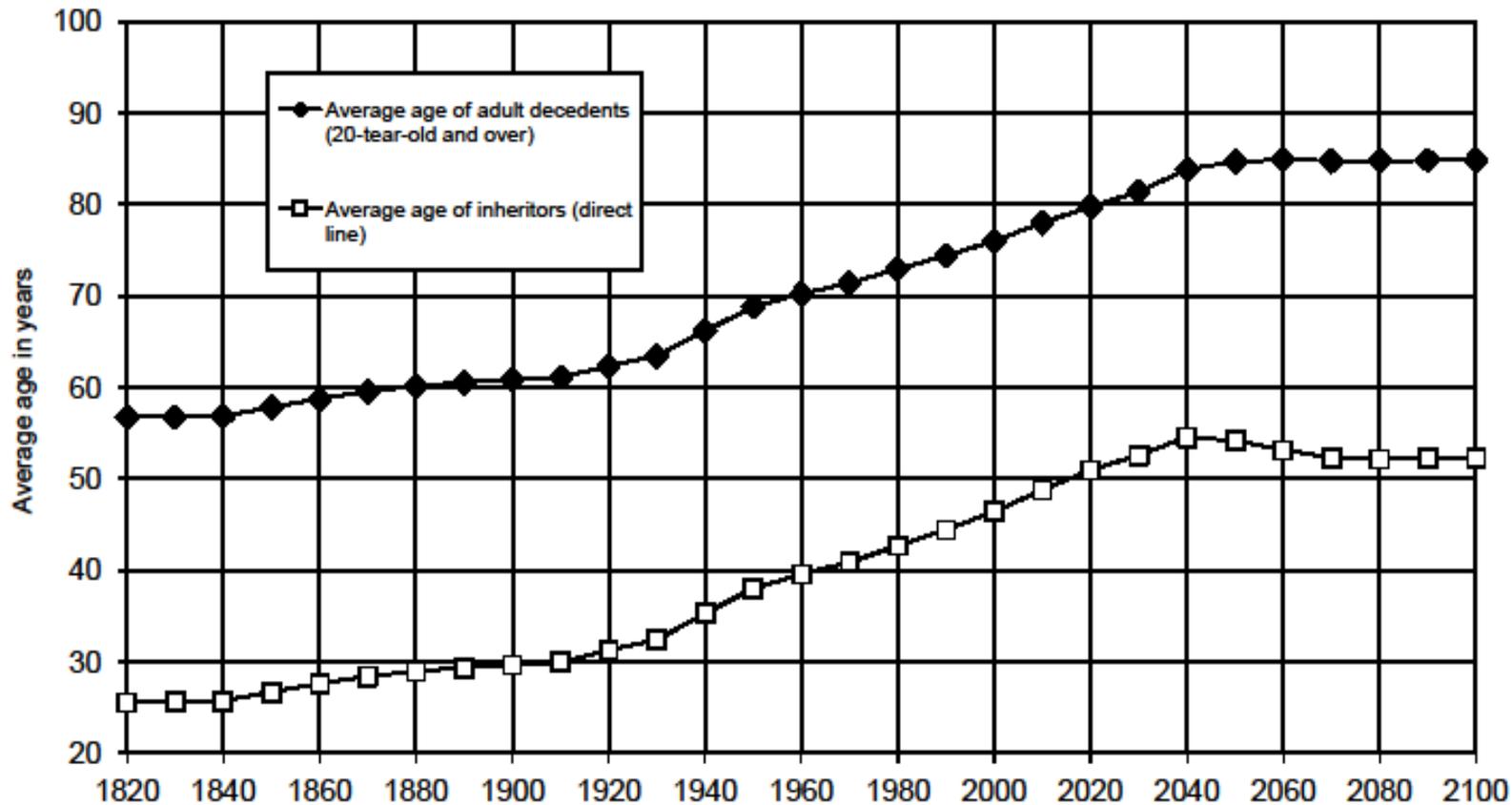
The annual inheritance flow was about 20-25% of national income during the 19th century and until 1914; it then fell to less than 5% in the 1950s, and returned to about 15% in 2010. Sources and series: see piketty.pse.ens.fr/capital21c.

Figure 11.2. The mortality rate in France 1820-2100



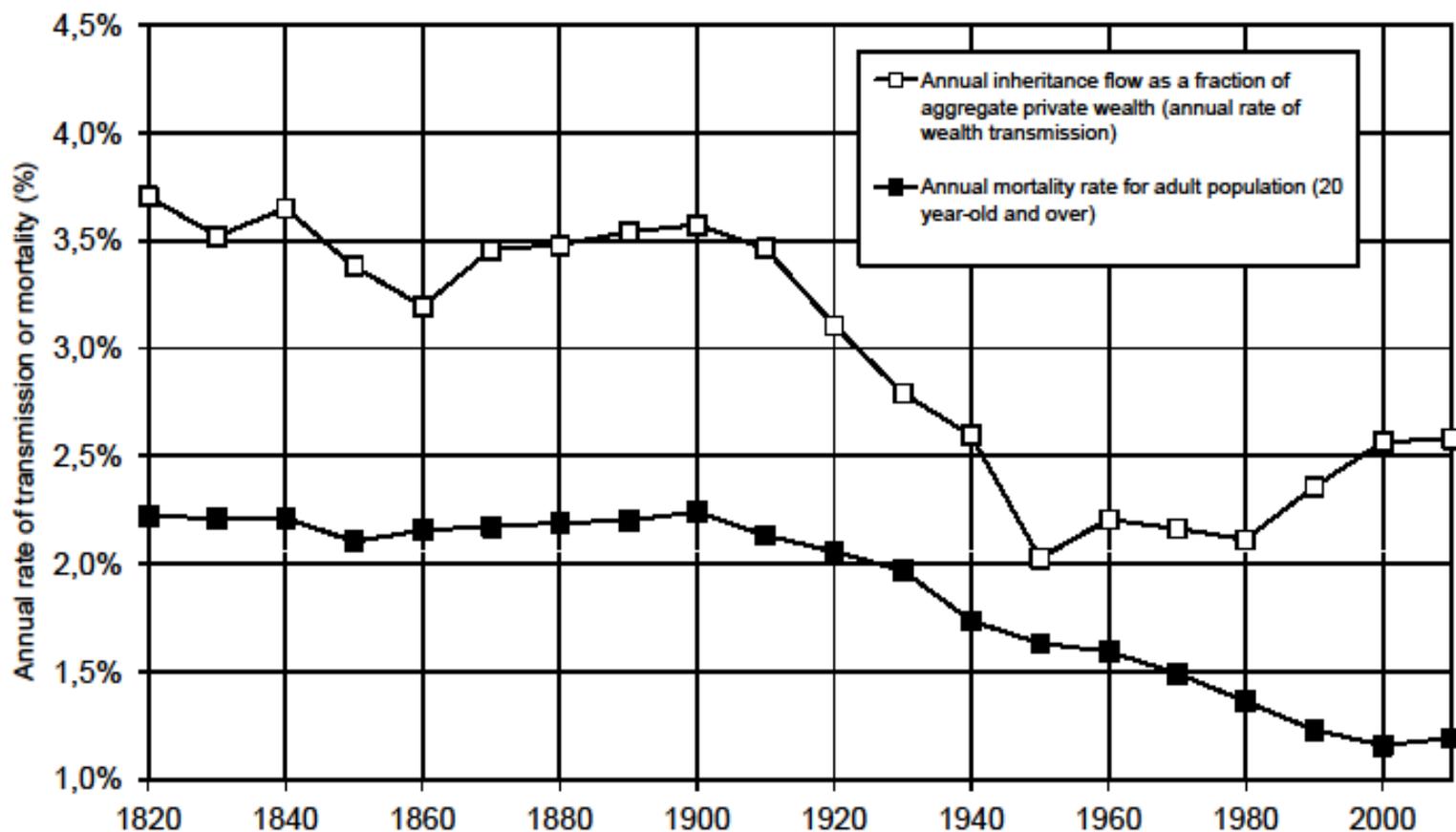
The mortality rate fell in France during the 20th century (rise of life expectancy), and should increase somewhat during the 21st century (baby-boom effect). Sources and series: see piketty.pse.ens.fr/capital21c.

Figure 11.3. Average age of decedents and inheritors, France 1820-2100



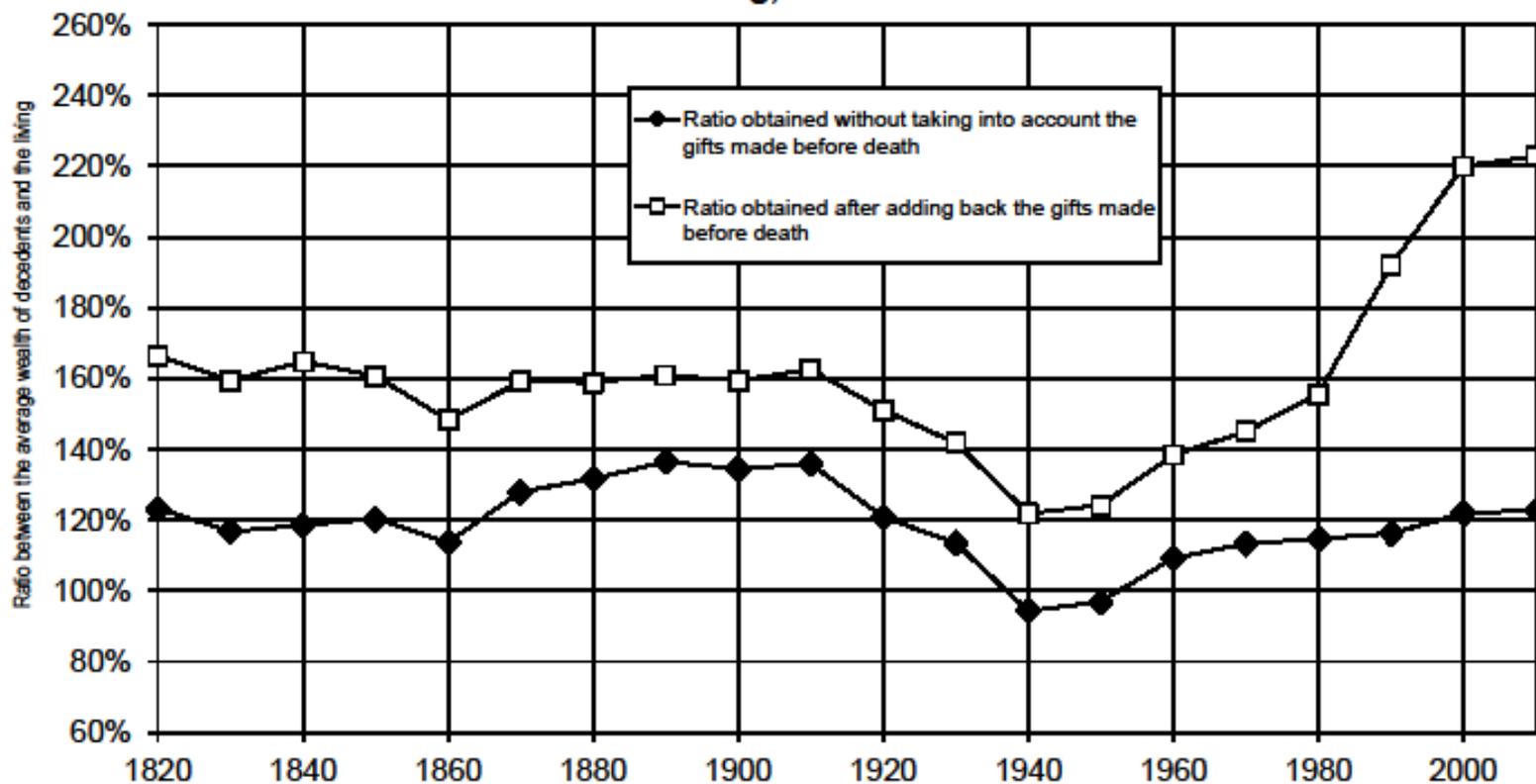
The average of (adult) decedents rose from less than 60 years to almost 80 years during the 20th century, and the average age at the time of inheritance rose from 30 years to 50 years. Sources and series: see piketty.pse.ens.fr/capital21c.

Figure 11.4. Inheritance flow vs. mortality rate, France 1820-2010



The annual flow of inheritance (bequests and gifts) is equal to about 2,5% of aggregate wealth in 2000-2010, vs. 1,2% for the mortality rate. Sources and series: see piketty.pse.ens.fr/capital21c

Figure 11.5. The ratio between average wealth at death and average wealth of the living, France 1820-2010



In 2000-2010, the average wealth at death is 20% higher than that of the living if one omits the gifts that were made before death, but more than twice as large if one re-integrates gifts. Sources and series: see piketty.pse.ens.fr/capital21c

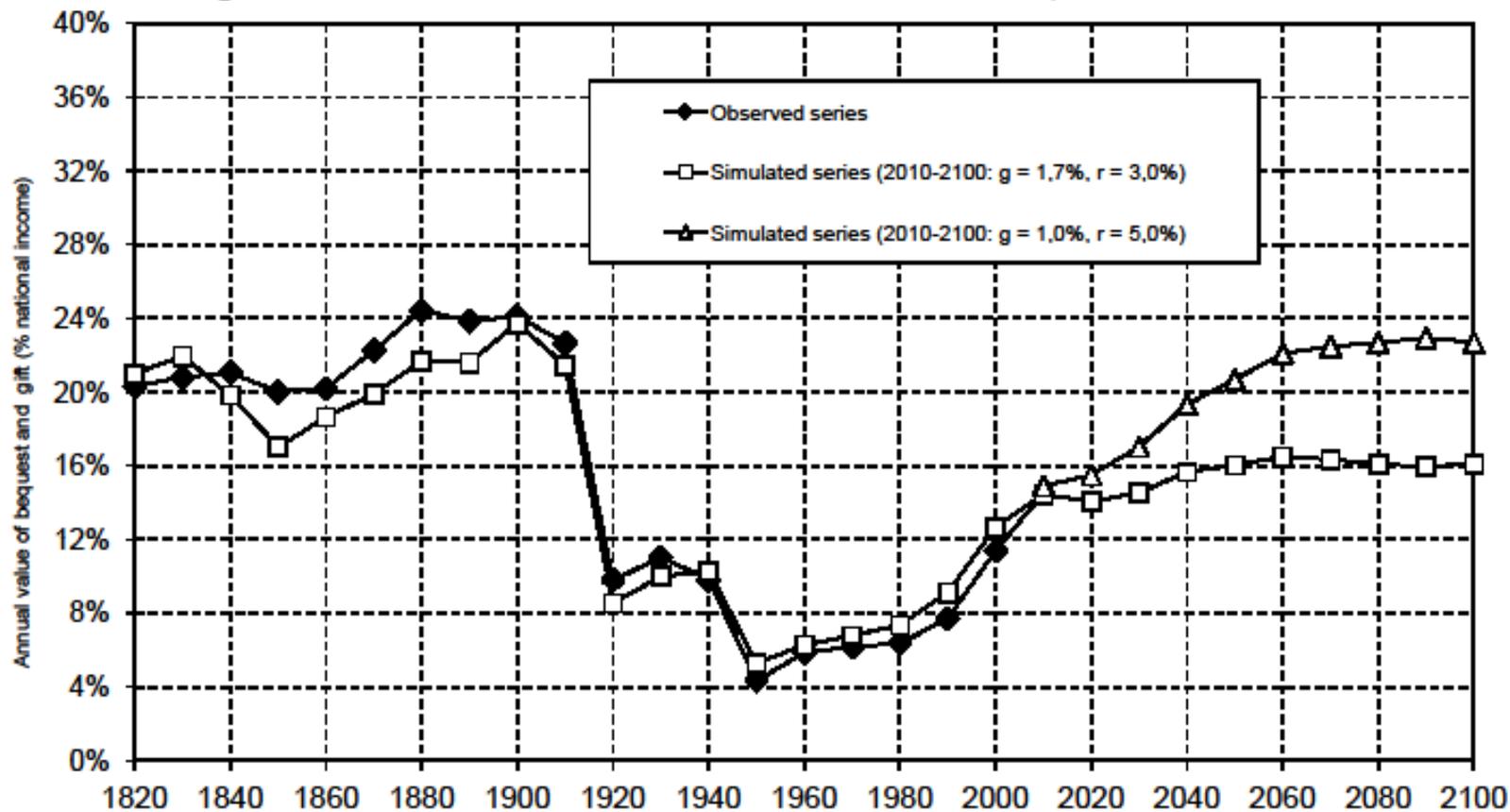
Table 11.1. The age-wealth profile in France, 1820-2010

Average wealth of each age group (% of average wealth of 50-59 year-old)	20-29 year	30-39 year	40-49 year	50-59 year	60-69 year	70-79 year	80 year and over
1820	29%	37%	47%	100%	134%	148%	153%
1850	28%	37%	52%	100%	128%	144%	142%
1880	30%	39%	61%	100%	148%	166%	220%
1902	26%	57%	65%	100%	172%	176%	238%
1912	23%	54%	72%	100%	158%	178%	257%
1931	22%	59%	77%	100%	123%	137%	143%
1947	23%	52%	77%	100%	99%	76%	62%
1960	28%	52%	74%	100%	110%	101%	87%
1984	19%	55%	83%	100%	118%	113%	105%
2000	19%	46%	66%	100%	122%	121%	118%
2010	25%	42%	74%	100%	111%	106%	134%

In 1820, the average wealth of individuals aged 60 to 69 was 34% higher than that of 50-to-59 year-old, and the average wealth of those aged 80 and over was 53% higher than that of 50-to-59 year old. Sources: see piketty.pse.ens.fr/capital21c.

- How is steady-state μ determined?
- One can show that for given saving behavior, μ is an increasing function of $r - g$
- Intuition: higher $r - g$ makes inheritance more important; young workers are relatively poor until they inherit; in aging societies, wealth also tends to get older and older, so that the rise of μ tends to compensate the decline in m
- As $g \rightarrow 0$, $\mu \rightarrow (D-A)/H$, so that $\mu \times m \rightarrow 1/H$
(with D = age at death, A = age at adulthood, H = generation length ≈ 30 years: $b_y = \mu \times m \times \beta \rightarrow \beta/H \approx 20\%$ if $\beta \approx 600$)
- Simple simulations: by assuming constant average saving rates by age group, one can replicate relatively well the 1810-2010 evolution of the aggregate inheritance flow in France; the future depends very much on $r - g$

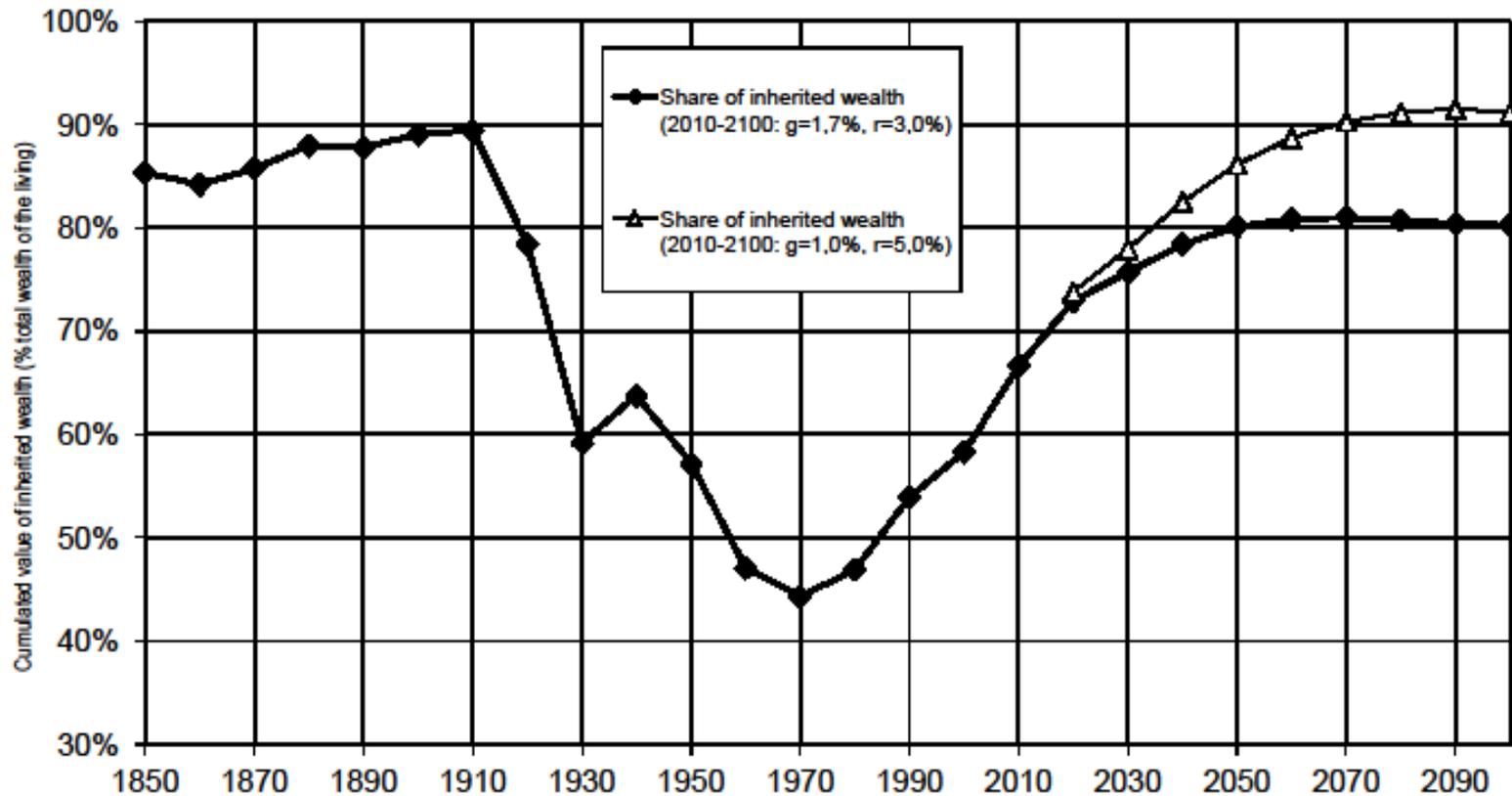
Figure 11.6. Observed and simulated inheritance flow, France 1820-2100



Simulations based upon the theoretical model indicate that the level of the inheritance flow in the 21st century will depend upon the growth rate and the net rate of return to capital. Sources and series: see piketty.pse.ens.fr/capital21c.

- In order to go from the annual flow of inheritance $b_y = B/Y$ to the share of the cumulated stock of inheritance in aggregate wealth $\phi = W_B/W$, one needs dynamic, individual-level data
- Simplified definition of ϕ : compare inheritance flow $b_y = B/Y$ with saving rate $s = S/Y$;
- If $s \approx 10\%$ and $b_y \approx 5\%$, then self-made wealth dominates inherited wealth (=mid 20c period)
- But if $s \approx 10\%$ and $b_y \approx 20\%$, then inherited wealth dominates self-made wealth (19c and 21c)
- T. Piketty, G. Zucman, “[Wealth and Inheritance in the Long-Run](#)”, Handbook of Income Distribution, 2014

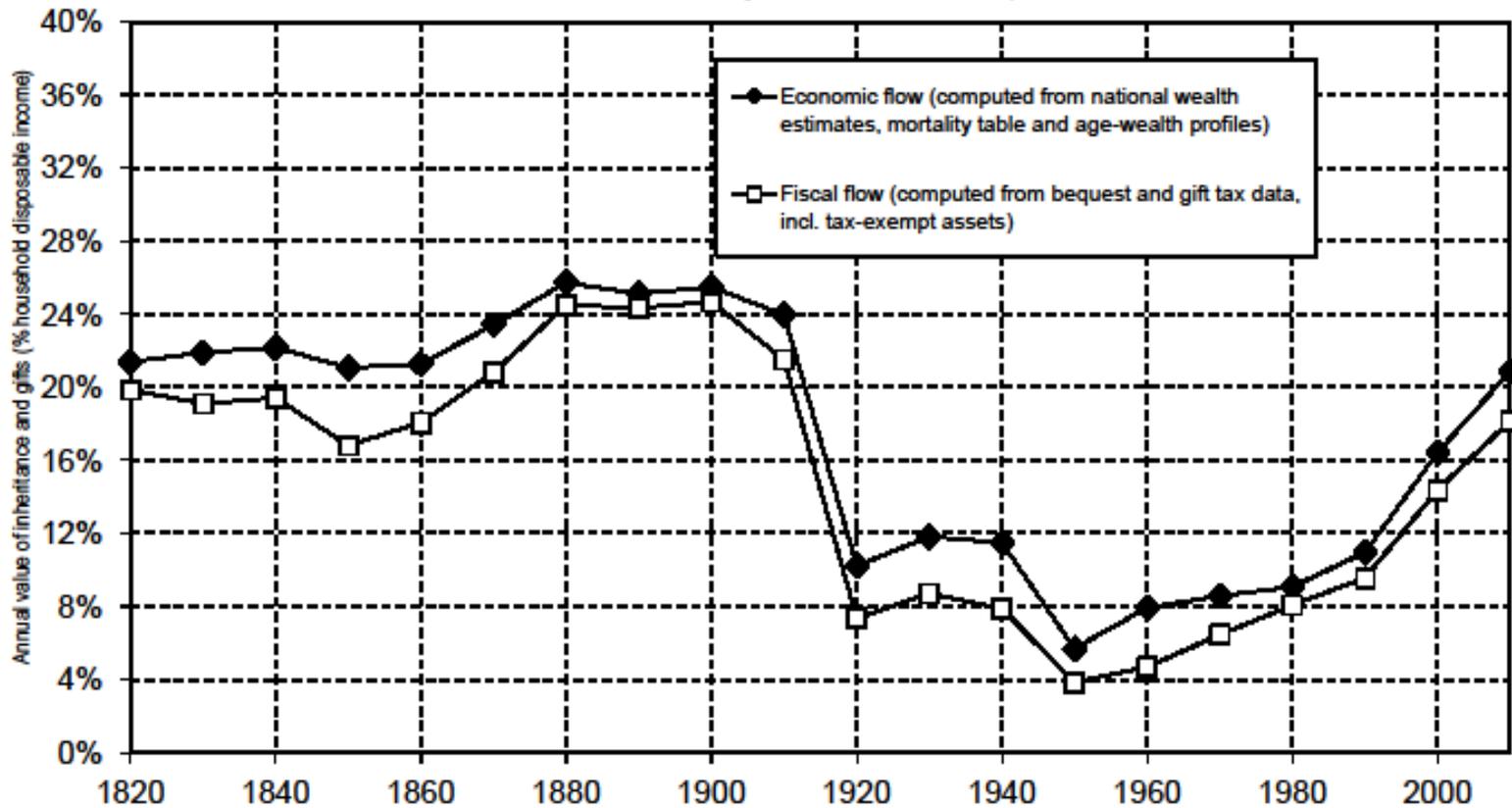
Figure 11.7. The share of inherited wealth in total wealth, France 1850-2100



Inherited wealth represents 80-90% of total wealth in France in the 19th century; this share fell to 40%-50% during the 20th century, and might return to 80%-90% during the 21st century. Sources and series: see piketty.pse.ens.fr/capital21c

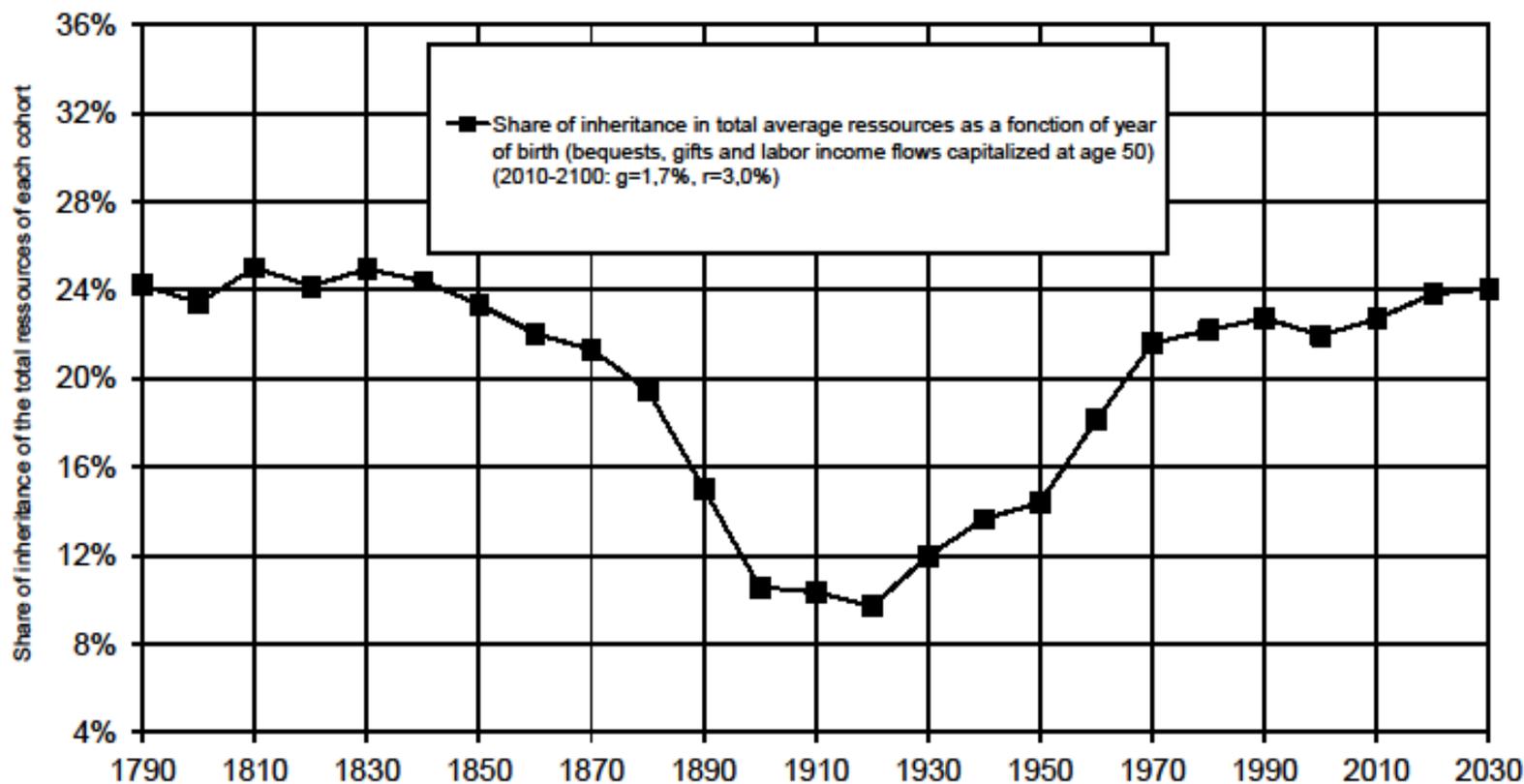
- Aggregate inheritance is (almost) back to 19c levels
- But the concentration of inheritance is not back to 19c levels: the bottom 50% is as poor as before, but the middle 40% now owns 20-30% of W
- Today, there are fewer very large inheritors than in 19c; and there are more large and middle-large inheritors
- This is a class model that is less unequal in some ways and more unequal in others; it is novel form of inequality, more merit-based in some ways, and more violent for losers in others

Figure 11.8. The annual inheritance flow as a fraction of household disposable income, France 1820-2010



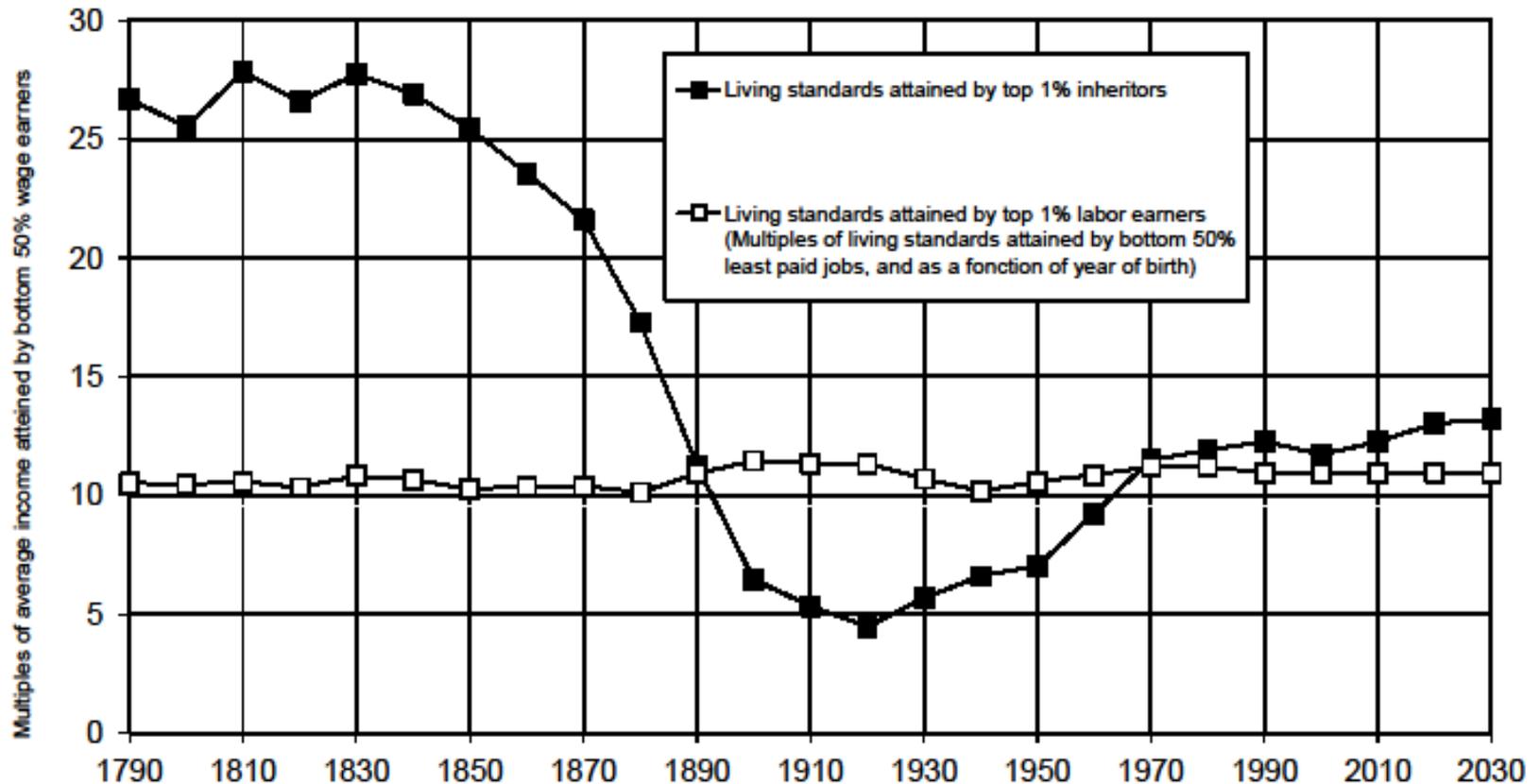
Expressed as a fraction of household disposable income (rather than national income), the annual inheritance flow is about 20% in 2010, i.e. close to its 19th century level. Sources and series: see piketty.pse.ens.fr/capital21c.

Figure 11.9. The share of inheritance in the total resources (inheritance and work) of cohorts born in 1790-2030



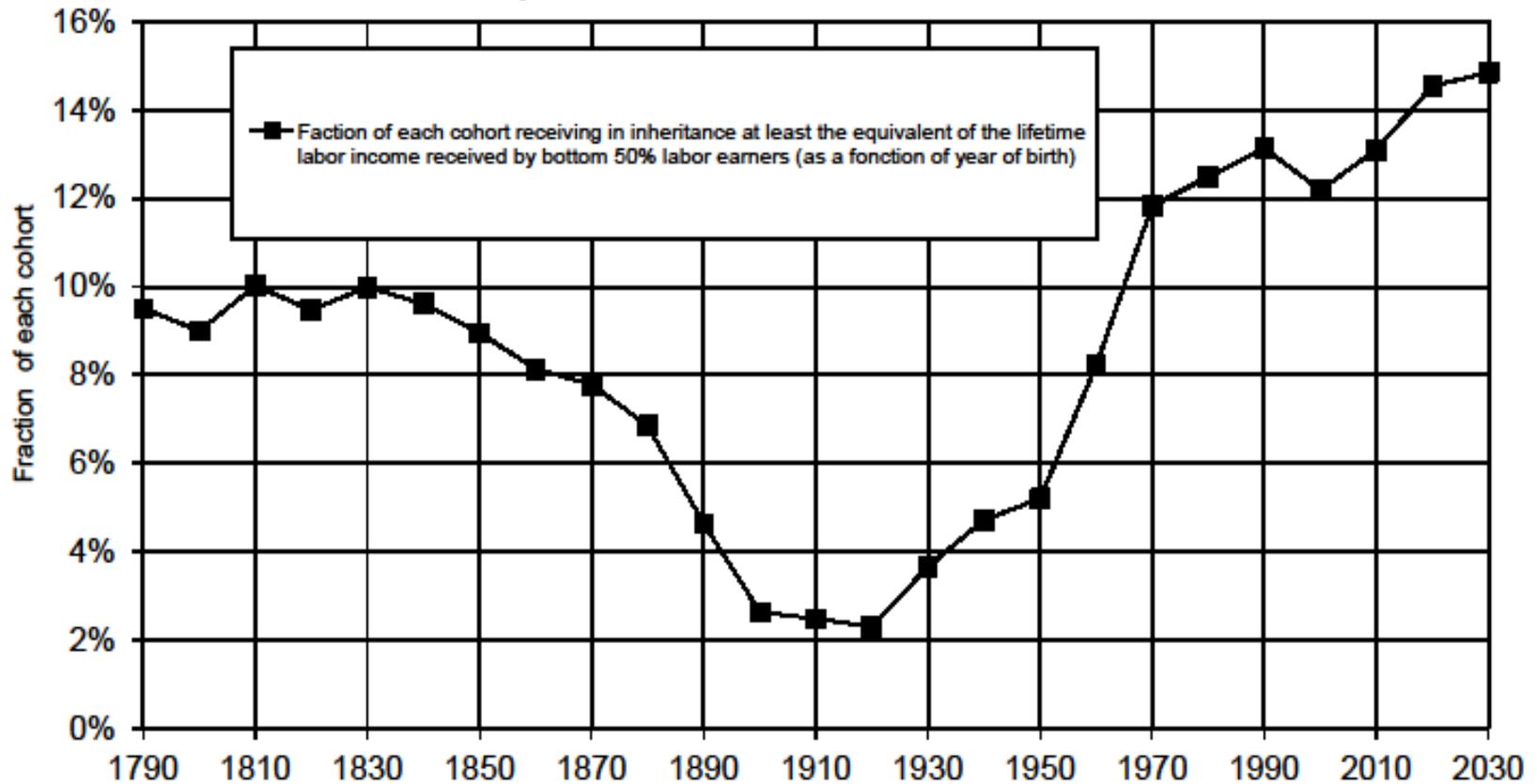
Inheritance made about 25% of the resources of 19th century cohorts, down to less than 10% for cohorts born in 1910-1920 (who should have inherited in 1950-1980). Sources and series: see piketty.pse.ens.fr/capital21c.

**Figure 11.10. The dilemma of Rastignac
for cohorts born in years 1790-2030**



In the 19th century, the living standards that could be attained by the top 1% inheritors were a lot higher than those that could be attained by the top 1% labor earners. Sources and series: see piketty.pse.ens.fr/capital21c.

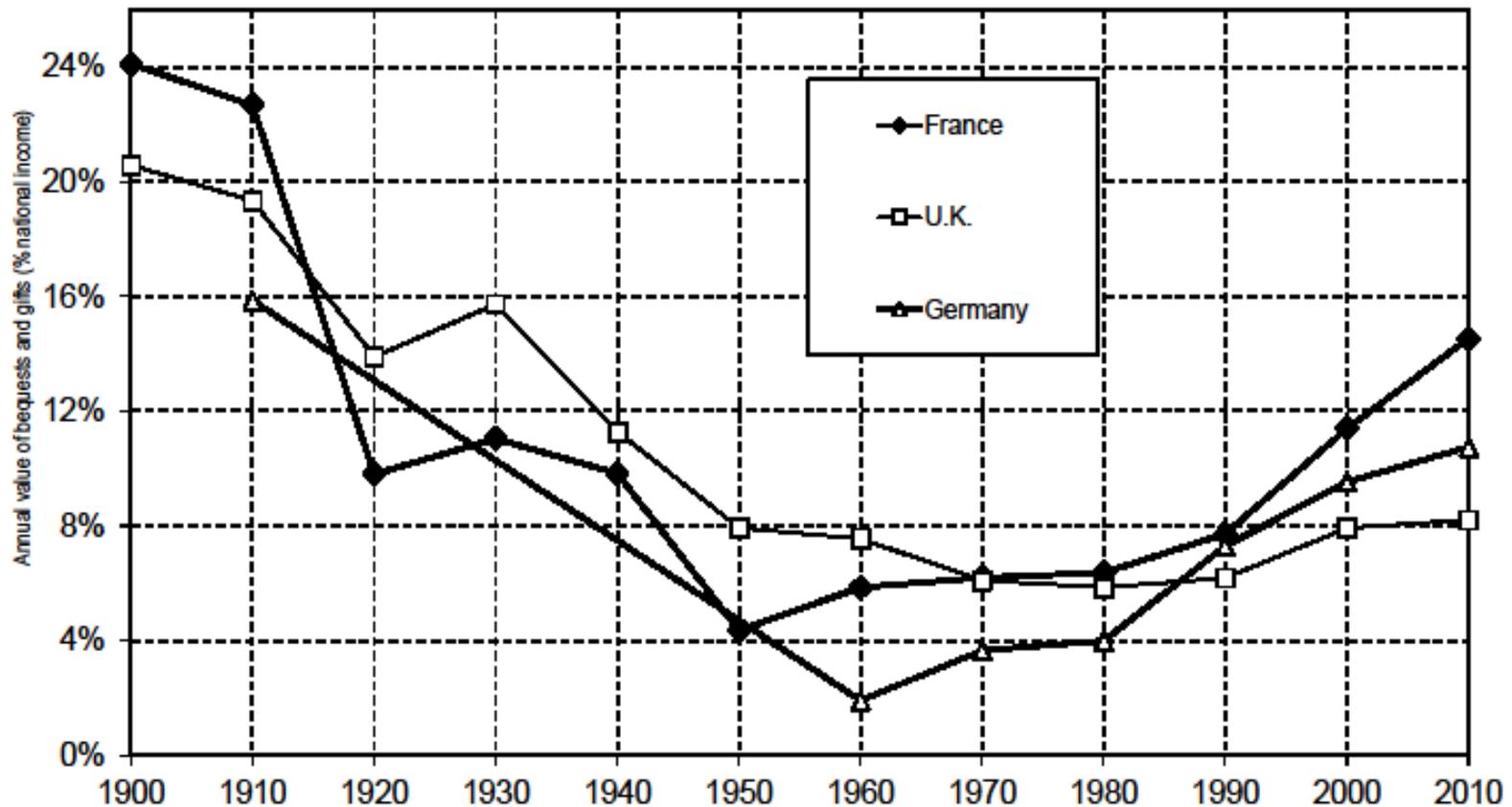
Figure 11.11. Which fraction of a cohort receives in inheritance the equivalent of a lifetime labor income?



Within the cohorts born around 1970-1980, 12-14% of individuals receive in inheritance the equivalent of the lifetime labor income received by the bottom 50% less well paid workers. Sources and series : see piketty.pse.ens.fr/capital21c

- Existing international evidence (Germany, UK, Sweden,...) suggests that France is relatively representative of what might be happening in other countries; and possibly in the entire world in the very long run if g decline everywhere

Figure 11.12. The inheritance flow in Europe 1900-2010



The inheritance flow follows a U-shaped in curve in France as well as in the U.K. and Germany. It is possible that gifts are underestimated in the U.K. at the end of the period. Sources and series: see piketty.pse.ens.fr/capital21c.