Economic Growth and the Elasticity of Substitution: Two Theorems and Some Suggestions

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At the beginning of the twentieth century, the work of the Swedish economist Knut Wicksell (1901) marked an important advance in our understanding of the relationship between the structure of production functions and income distribution. Wicksell had asked the following question, among many others: Suppose that the marginal productivities are equal to the remuneration rates of each factor. What then should be the form of the corresponding production function? He pointed out (p. 125) that the general integral of the partial differential equation implied by this relationship was of the form $Y = Lf(K/L)$, which meant in turn that the product Y was necessarily a homogeneous of degree one function of capital (K) and labor (L). As an example, he gave a function where the shares of K and L in Y were constant and wrote down the function which would be rediscovered decades later by Charles W. Cobb and Paul H. Douglas (1928).[1]

However, it is only half a century later, in Robert M. Solow's Nobel Prize-winning 1956 essay, that implications between the structure of the production function and long-run equilibrium or disequilibrium of the economy were drawn. In his classic article " A Contribution to the Theory of Economic Growth," Solow had considered three cases analytically (other cases were also discussed geometrically, using a phase diagram). He had labeled the two first cases "Harrod-Domar" and "Cobb-Douglas," respectively, and he referred to the third type of production function as "offering a bit of variety"; it was $Y = (aK^{1/2} + L^{1/2})^2$, a particular case of $Y = (aK^p + L^p)^{1/p}$. In those early days, Solow had not pointed out to the reader that his cases could have been numbered 0, 1, and 2, following the elasticity of substitution of the corresponding production function. It would be only five years later, in 1961, that with his coauthors Kenneth J. Arrow et al., he would find the general form of the two-factor constant-elasticity-of-substitution (CES) production function by integrating the differential equation corresponding to the observed relationship between output per capita and the wage rate.

Solow's third case was indeed highly interesting since it allowed the possibility of ever-increasing income per head, even without any technical progress. Indeed, Solow showed that due to the efficiency of the system entailed by the particular production function he had chosen, there existed a threshold for the savings rates s to generate investment per head large enough for income per head to grow forever. That threshold was $s = n/a^2$ where n is the growth rate of the population. It was then natural to ask whether one could express that threshold in terms of the elasticity of substitution σ.

The value of that threshold was determined by de La Grandville (1989) to be equal to $s = n\beta(\sigma)^{\sigma/(1-\sigma)}$, where $\beta(\sigma)$ is K's coefficient in a normalized CES function (equal to a in Solow's third case, where $\sigma = 2$); for any s and n values, this relationship gives also the threshold value $\hat{\sigma}$ in implicit form by the solution of $\sigma = 1/[1 - \log \beta(\sigma)/\log(n/s)]$. By totally differentiating the last relationship, one could observe that the threshold for the elasticity of substitution to entail ever-sustained growth was an in-

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[1] Wicksell, Cobb, and Douglas would no doubt burst with pride if they knew that one of our first-year students wrote, in a paper of his: "Of course, we recognize here a Kirk Douglas production function."

creasing function of the population growth rate, and a decreasing function of the saving-investment coefficient. Furthermore, de La Grandville showed, using a geometrical argument,[2] that even if the threshold for s (equivalently, for σ, given s and n) was not reached, a higher elasticity of substitution entailed both a higher growth rate of income per head, and a higher steady-state value of income per head. He concluded that the elasticity of substitution may be a powerful engine of growth, and made the conjecture that miracle growth in Japan and East Asian countries was not necessarily due to a higher savings rate or more efficient technical progress, but to a higher elasticity of substitution. To the best of our knowledge, this hypothesis was tested twice: Ky-Hyang Yuhn (1991) confirmed it in the case of South Korea; it was also successfully tested by Francis D. Cronin et al. (1997) in the case of the U.S. telecommunications industry.

Our purpose in this paper is twofold: first, we will prove two theorems. We will show that (1) when two countries start from common initial conditions, the one with the higher elasticity of substitution will always experience, other things being equal, a higher income per head; (2) any equilibrium values of capital-labor and income per head are increasing functions of σ. The analytical argument will extend earlier work by Klump (1997, 1998) on the formal properties of normalized CES functions. Our second objective is to make a number of suggestions: first, we would like to advocate the use of normalized CES functions in growth models; second, we propose that empirical research in this area be based upon the "Morishima-Blackorby-Russell" elasticity of substitution; and finally, we suggest some methods to determine the functional form of increasing-elasticity-of-substitution production functions.

[2] The geometrical argument ran as follows: let the family of all homogeneous of degree-one production functions $F_\sigma(K, L)$ (dependent on the elasticity of substitution σ), be surfaces which are normalized in the sense that they are tangent to each other along a ray from the origin in the three-dimensional space (Y, K, L); this ray projects itself on the (K, L) plane along ray $K = \bar{k}L$, where $\bar{k}$ is arbitrarily chosen; in two-dimensional space (K, L), this normalization corresponds to the familiar point of common tangency of all isoquants $Y_0 = F_\sigma(K, L)$ for a given level of production Y_0. When σ increases from 0 to infinity, the isoquants (initially L-shaped) become flatter and flatter at their common tangency point; in the limit, when $\sigma \to \infty$, they become a straight line. In three-dimensional space, the surface opens up like a chrysalid; from an edge equal to $\min[K/c_1, L/c_2]$ when $\sigma = 0$, with $c_1/c_2 = \bar{k}$, it unfolds unto a smooth concave surface and finally unto a plane through the origin, when $\sigma \to \infty$.

Now the fact that the production function is homogeneous of degree one enables to write output (or income) per head as $Y/L = y = F_\sigma(K/L, 1) = f_\sigma(k)$; $F_\sigma(K/L, 1)$ is also equal to the family of the vertical sections of the surfaces at point $L = 1$, i.e., $F_\sigma(K/L, 1) = F_\sigma(k, 1)$. Thus, in (Y, K) space, and equivalently in (y, k) space, the opening up of the $F_\sigma(K, L)$ surfaces around a ray from the origin translates into the opening up of the kinked line corresponding to the Walras-Leontief function, around its kink at $k = \bar{k}$. If one then considers the Solow phase diagram, the effect of this is to open up everywhere (except at the common tangency point) the $sf(k)$ function, which is the actual investment per head function (see Figure 1). In turn, this increases for every k the right-hand side of Solow's equation of motion $\dot{k} = sf_\sigma(k) - nk$.

I. Two Theorems

We consider a neoclassical growth model in the tradition of Solow (1956). Per capita production and income are expressed by a homogeneous of degree one, constant-elasticity-of-substitution production function $y = A[ak^\psi + (1 - a)]^{1/\psi}$ as it was first proposed by Arrow et al. (1961), with $\psi \equiv \sigma/(\sigma - 1)$. We are interested in comparing economies which are distinguished by their elasticity of substitution σ only. Therefore we normalize the CES function, as de La Grandville (1989) did, at some arbitrarily chosen baseline values for the three following variables: the capital-labor ratio (or capital intensity) $\bar{k}$, the per capita production (or per capita income) $\bar{y} = f(\bar{k})$, and the marginal rate of substitution $\bar{m} = [f(\bar{k}) - \bar{k}f'(\bar{k})]/f'(\bar{k})$. As a first step we obtain a system of equations in a and A:

$$\bar{m} = \frac{(1 - a)}{a} \cdot \bar{k}^{1-\psi} \tag{1}$$

$$\bar{y} = A[a\bar{k}^\psi + (1 - a)]^{1/\psi}, \tag{2}$$

from which we can then deduce the value of these parameters as functions of σ, given $\bar{k}$, $\bar{y}$, and $\bar{m}$:

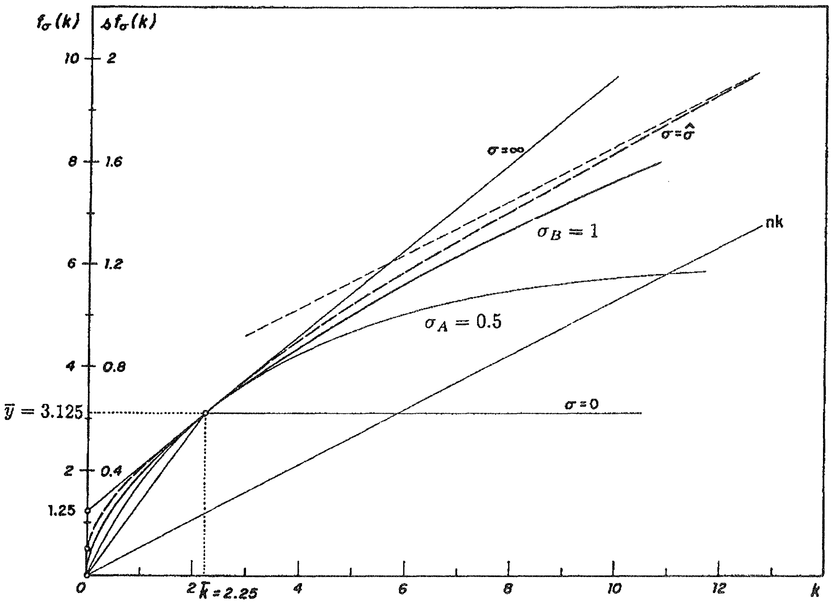

FIGURE 1. THE RIGHT-HAND SIDE OF THE EQUATION OF MOTION $\dot{k} = sf_\sigma(k) - nk$

Note: For any value of the capital-labor ratio $k(k \neq \bar{k})$, income per head and the rate of increase of k are larger in country B than in country A.

$$(3) \quad a = \frac{\bar{k}^{1-\psi}}{\bar{k}^{1-\psi} + \bar{m}} = a(\sigma;\, \bar{k},\, \bar{m}) \equiv a(\sigma)$$

$$(4) \quad A = \bar{y}\left(\frac{\bar{k}^{1-\psi} + \bar{m}}{\bar{k} + \bar{m}}\right)^{1/\psi}$$

$$= A(\sigma;\, \bar{k},\, \bar{m},\, \bar{y}) \equiv A(\sigma).$$

The normalized CES function can thus be written as:

$$(5) \quad y = f_\sigma(k)$$

$$= A(\sigma)\{a(\sigma)k^\psi + [1 - a(\sigma)]\}^{1/\psi}.^3$$

From equation (5), taking into account expressions (3) and (4), we can calculate the profit share π as:

$$(6) \quad \pi = \frac{kf'_\sigma(k)}{f_\sigma(k)} = \frac{ak^\psi}{ak^\psi + (1-a)}$$

$$= a\left[\frac{Ak}{f_\sigma(k)}\right]^\psi$$

$$= \frac{k^\psi}{k^\psi + (1-a)/a} = \frac{k^\psi \bar{k}^{1-\psi}}{k^\psi \bar{k}^{1-\psi} + \bar{m}}.$$

It should be stressed here that for $k \neq \bar{k}$ the profit share π depends on σ as well as on k, whereas the baseline profit share $\bar{\pi}$ (for $k = \bar{k}$):

$$(7) \quad \bar{\pi} = \frac{\bar{k} \cdot f_\sigma(\bar{k})}{f_\sigma(\bar{k})} = a\left[\frac{A\bar{k}}{\bar{y}}\right]^\psi = \frac{\bar{k}}{\bar{k} + \bar{m}}$$

is independent from both.

Equations (6) and (7), together with equations (3) and (4), are now used to reformulate the normalized CES production function (5) in the following way (see Appendix A for the detailed calculations):

[3] These parametric curves are all tangent to each other at point $\bar{k}$; they unfold from a kinked line into a straight line (the common tangent) when σ increases from 0 to infinity (see Figure 1).

where we have made use of the right-hand side of the equation of motion (9) to replace $\dot{k}$. Since both $f_\sigma(k)$ and π (for $k > \bar{k}$) are increasing functions of σ, we get the above-mentioned result.

Consider now the case where a steady state exists. This implies that the elasticity of substitution does not reach the threshold value indicated in our introduction. We are then able to prove our second theorem.

THEOREM 2: *Suppose that two economies are described by CES production functions differing only by their elasticity of substitution, and share initially a common capital-labor ratio ($\bar{k}$), population growth rate (n), and saving-investment rate (s). If the levels of the elasticities of substitution guarantee the existence of steady states, then the economy with the higher elasticity of substitution will have a higher capital intensity and a higher level of per capita income in the steady state.*

PROOF:

From the equation of motion (9) we obtain the steady-state capital intensity k^*:

$$k^* = \left[\frac{1-a}{\left(\frac{n}{sA}\right)^{\psi} - a}\right]^{1/\psi} \tag{15}$$

as well as the following expression for the steady-state profit share π^* (for $k = k^*$):

$$\pi^* = \frac{k^* \cdot f'_\sigma(k^*)}{f_\sigma(k^*)} = a\left[\frac{Ak^*}{f_\sigma(k^*)}\right]^{\psi} = \frac{\bar{k}^{1-\psi}}{\bar{k} + \bar{m}} \cdot \left(\frac{s\bar{y}}{n}\right)^{\psi}. \tag{16}$$

Existence and stability of the steady state require that both factors are essential for production and thus both must receive a share of total income. This implies that $0 < \pi^* < 1$.

The steady-state profit share, which is a function of σ, can be used to rewrite k^* in the following way:

$$k^* = \left[\frac{(1-\bar{\pi})}{\bar{\pi}} \cdot \frac{\pi^*}{(1-\pi^*)}\right]^{1/\psi} \cdot \bar{k} \tag{17}$$

[see Appendix C for the detailed derivations of equation (17) and the subsequent equations (18) and (19)]. It should be noted here again that the baseline profit share $\bar{\pi}$, as it was defined in equation (7), is independent from σ. We can now calculate the derivative of the steady-state profit share with respect to σ as:

$$\frac{\partial \pi^*}{\partial \sigma} = \frac{1}{\psi} \cdot \frac{1}{\sigma^2} \cdot \pi^* \ln\left(\frac{\pi^*}{\bar{\pi}}\right). \tag{18}$$

Hence from (17) the influence of σ on k^* can be derived as:

$$\frac{\partial k^*}{\partial \sigma} = -\frac{1}{\sigma^2} \cdot \frac{1}{\psi^2} \cdot \frac{k^*}{(1-\pi^*)} \times \left[\pi^* \ln\left(\frac{\bar{\pi}}{\pi^*}\right) + (1-\pi^*)\ln\left(\frac{1-\bar{\pi}}{1-\pi^*}\right)\right]. \tag{19}$$

Again making use of the strict concavity of the logarithmic function, we have the two inequalities:

$$\ln\left(\frac{\bar{\pi}}{\pi^*}\right) < \frac{\bar{\pi}}{\pi^*} - 1 \text{ and} \qquad \ln\left(\frac{1-\bar{\pi}}{1-\pi^*}\right) < \frac{1-\bar{\pi}}{1-\pi^*} - 1. \tag{20}$$

Multiplying by π^* and by $(1 - \pi^*)$, respectively, and then adding up yields:

$$\pi^* \ln\left(\frac{\bar{\pi}}{\pi^*}\right) + (1-\pi^*)\ln\left(\frac{1-\bar{\pi}}{1-\pi^*}\right) < \bar{\pi} - \pi^* + \pi^* - \bar{\pi} = 0. \tag{21}$$

From (19) and (21) it follows that $\partial k^*/\partial\sigma > 0$. Also, since per capita income in the steady state is a positive function of both σ and k^*, it must be higher in the country with the higher elasticity of substitution.

Both theorems are illustrated in Figure 1, where we have chosen to draw three CES functions, with $\sigma = 0$, $\sigma = 0.5$, and $\sigma = 1$, respectively.

II. Some Suggestions

A. *Normalized CES Functions in Growth Theory*

The use of a normalized CES production function enabled us to work with a family of functions which are distinguished by their constant elasticity of substitution only. Their commonly shared values $\bar{k}$, $\bar{y}$, and $\bar{m}$, together with σ (or the substitution parameter ψ) determine the parameters a and A of the standard CES function proposed by Arrow et al. (1961). It should be noted that such a family of CES functions also includes the following normalized special cases: these are, for $\sigma = 0$, the Walras-Leontief function $y = \bar{y}\min[k/\bar{k}, 1]$; for $\sigma = 1$, the Cobb-Douglas function $y = \bar{y}(k/\bar{k})^{\bar{k}/(\bar{k}+\bar{m})}$; and for $\sigma \rightarrow \infty$, the normalized linear function $y = [\bar{y}/(\bar{k} + \bar{m})](\bar{m} + k)$.

We suggest that in growth models the use of a normalized standard CES function is preferable to a new functional form as the one recently proposed by Robert J. Barro and Xavier Sala-i-Martin (1995 pp. 43–49). These authors introduce a CES production function of the form:

$$y = f(k) = A[a(bk)^{\psi} + (1 - a)(1 - b)^{\psi}]^{1/\psi},$$

where $0<b<1$, in a traditional Solow growth model and then generate the Harrod-Domar model as a special case for $\sigma = 0$. They justify their new parameter b by pointing out an alleged inconvenience of the standard CES function for σ approaching zero. This inconvenient property of obtaining identical income shares for both factors of production in that limit case can only occur, however, as long as the parameters a and A are assumed to be independent of σ. With normalized standard CES functions this problem is avoided. And if our normalization procedure is applied to the Barro-Sala-i-Martin form the (somehow curious) parameter b vanishes. Klump and Harald Preissler (2000) discuss other functional forms of the CES function which have been used in models of economic growth and show how they can be transformed into the normalized standard form.

B. *Empirical Testing*

As we mentioned earlier, the elasticity of substitution, as an engine of growth, seems to have been successfully tested by Yuhn (1991) and Cronin et al. (1997). However, we take the liberty of submitting a suggestion that might be useful to those who would be willing to test further this partial explanation of the growth process.

To the best of our knowledge, any test of this conjecture was carried out in a multiple-factor model that used the concept of the Allen-Uzawa partial elasticity of substitution. As Charles Blackorby and Robert R. Russell (1989) have convincingly demonstrated, the Allen-Uzawa concept suffers from rather serious shortcomings: it is not a complete measure of the ease of substitution; it does not provide any information about income distribution among factors; and finally it cannot be interpreted as the relative change of an input ratio to a price ratio.

On the other hand, another concept of the partial elasticity of substitution was developed independently by Michio Morishima, in a paper published in Japanese only (1967), and by Blackorby and Russell (1975). This concept does not carry the above-mentioned deficiencies. It seems to us that further tests of our conjecture should use the Morishima-Blackorby-Russell elasticity of substitution, as one of us suggested to call it (see de La Grandville, 1997), rather than the Allen-Uzawa one.

C. *Towards Production Functions with Increasing Elasticity of Substitution*

Finally, we should reflect on the following question, which takes us back to the very birth of the CES function. Let us recall that Arrow et al. (1961) had observed that the empirical relationship between income per head (y) and the wage rate (w) was not compatible with a production function such as the Cobb-Douglas. They had observed, and successfully tested, a power function of the type $y = cw^{\sigma}$, with σ (the elasticity of y with respect to w, also equal to the elasticity of substitution) significantly smaller than one. This was in contradiction with the Cobb-Douglas function whose elasticity of substitution equals one; hence the remarkable idea of deriving, through the integration of $y = cw^{\sigma} = c(y - ky')^{\sigma}$, the CES function. Notice

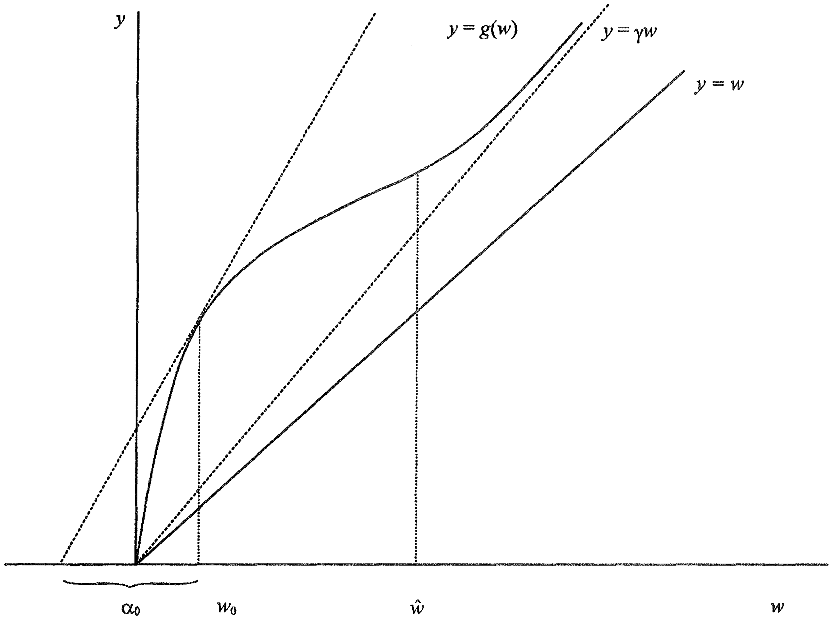

FIGURE 2. A POSSIBLE RELATIONSHIP BETWEEN INCOME PER HEAD y AND THE WAGE RATE w

Note: The elasticity of y with respect to w, equal to the elasticity of substitution ($=w_0/\alpha_0$ at w_0) ultimately increases and tends towards one.

now that $\sigma < 1$ implies an ever-increasing share of labor income in total income. Indeed, recall that $wL/Y = w/y$ is the share of labor income. Since $\sigma = (dy/y)/(dw/w) < 1$, it implies $dw/w > dy/y$ and therefore w/y increases; from equation (16) we can see that with $\sigma < 1$ (and therefore $\psi < 0$) the steady-state profit share is a decreasing function of the population rate of growth n, and an increasing function of s and A. In the limit, this share even tends to zero if n tends to zero. This comes from the fact that, with $\sigma < 1$, the marginal product of capital falls to zero, and output per head is bounded.

We therefore may wonder whether the elasticity of substitution would be an increasing function of k for large values of k (equivalently, for large values of w and y). We might then want to test statistically a function between w and y [denoted $y = g(w)$] with the following properties (see Figure 2): after an inflection point at $\hat{w}$ its curve would become convex, tending asymptotically towards a ray from the origin, $y = \gamma w$, with $\gamma > 1$. This implies that the elasticity of substitution [equal to $\sigma = (dy/dw) \cdot (w/y) = (y/\alpha) \cdot (w/y) = w/\alpha$ on Figure 2] would ultimately increase; asymptotically it would tend towards one. In a way, the resulting production function, solution of $y = g(y - ky')$, would have a form of increasing efficiency built in. Whenever sufficient conditions of growth are met (when the capital-labor ratio and income per head reach a certain level) the system would start shifting towards a more efficient one.

Another route, which may parallel the one described above, is to consider that the relationship between y and w is shifting through time, as would the resulting production function. We would have a relation of the form $y = h(w, t) = h(y - ky', t)$. If such a relationship is supported by strong statistical evidence, and if the corresponding differential equation is amenable to a closed-form solution, who knows? We might even learn something about the still-elusive, and somewhat mysterious, concept of technological progress.

There are hard questions to solve: the statistical dependencies might be difficult to interpret; also, the changes in the elasticity of substitution are dependent on the little-known third derivatives of the production function; and finally the differential equations just mentioned may not yield closed-form solution. Nevertheless, it seems more than appropriate to remember here that Arrow et al. (1961 p. 246) had suggested, in the conclusion of their essay, that "the process of economic development itself might shift the over-all elasticity of substitution." More than three decades later, we surmise that this is exactly what may have happened, and what may continue to happen in the future.

APPENDIX A: NORMALIZED CES FUNCTION AND INCOME DISTRIBUTION

As shown in equations (1)–(5), given the baseline values $\bar{k}$, $\bar{y}$, and $\bar{m}$ for capital intensity, per capita income, and the marginal rate of substitution, respectively, the normalized CES function:

$$\text{(A1)}\quad y = f_\sigma(k) = A(\sigma)\{a(\sigma)k^\psi + [1 - a(\sigma)]\}^{1/\psi},$$

with $\psi \equiv \sigma/(\sigma - 1)$, has the following parameters:

$$\text{(A2)}\quad a = \frac{\bar{k}^{1-\psi}}{\bar{k}^{1-\psi} + \bar{m}} = a(\sigma;\, \bar{k},\, \bar{m}) \equiv a(\sigma)$$

$$\text{(A3)}\quad A = \bar{y}\cdot\left(\frac{\bar{k}^{1-\psi} + \bar{m}}{\bar{k} + \bar{m}}\right)^{1/\psi} = A(\sigma;\, \bar{k},\, \bar{m},\, \bar{y}) \equiv A(\sigma).$$

We can calculate the profit share π (for $k \neq \bar{k}$) as:

$$\text{(A4)}\quad \pi = a\left[\frac{Ak}{f_\sigma(k)}\right]^\psi = \frac{k^\psi\bar{k}^{1-\psi}}{k^\psi\bar{k}^{1-\psi} + \bar{m}},$$

whereas the baseline profit share $\bar{\pi}$ (for $k = \bar{k}$) is given by:

$$\text{(A5)}\quad \bar{\pi} = a\left[\frac{A\bar{k}}{\bar{y}}\right]^\psi = \frac{\bar{k}}{\bar{k} + \bar{m}}.$$

(A4) and (A5) together imply the relationship:

$$\text{(A6)}\quad \frac{\pi}{\bar{\pi}}\cdot\frac{(1-\bar{\pi})}{(1-\pi)} = \frac{k^\psi\bar{k}^{1-\psi}}{\bar{m}}\cdot\frac{\bar{m}}{\bar{k}} = \left(\frac{k}{\bar{k}}\right)^\psi.$$

Making use of (A2), (A3), and (A6) we can reformulate the normalized CES function (A1) in the following way:

$$\text{(A7)}\quad y = f_\sigma(k) = \bar{y}\cdot\left(\frac{\bar{k}^{1-\psi} + \bar{m}}{\bar{k} + \bar{m}}\right)^{1/\psi} \times \left(\frac{k^\psi\bar{k}^{1-\psi} + \bar{m}}{\bar{k}^{1-\psi} + \bar{m}}\right)^{1/\psi}$$

$$= \bar{y}\cdot\left(\frac{k^\psi\bar{k}^{1-\psi} + \bar{m}}{\bar{k} + \bar{m}}\right)^{1/\psi}$$

$$= \bar{y}\cdot\left(\frac{1-\bar{\pi}}{1-\pi}\right)^{1/\psi}$$

$$= \frac{\bar{y}}{\bar{k}}\cdot\left(\frac{\bar{\pi}}{\pi}\right)^{1/\psi}\cdot k.$$

Equation (A7) corresponds to equation (8) in our text.

APPENDIX B: ELASTICITY OF SUBSTITUTION AND PER CAPITA INCOME

From (A4) we obtain for the derivative of the profit share π with respect to σ:

$$\text{(B1)}\quad \frac{\partial\pi}{\partial\sigma} = \frac{1}{\sigma^2}\cdot\frac{\partial\pi}{\partial\psi}$$

$$= \frac{1}{\sigma^2}\cdot\frac{1}{(k^\psi\bar{k}^{1-\psi} + \bar{m})^2} \times [k^\psi\bar{k}^{1-\psi}(k^\psi\bar{k}^{1-\psi} + \bar{m})(\ln k - \ln \bar{k}) - k^\psi\bar{k}^{1-\psi}k^\psi\bar{k}^{1-\psi}(\ln k - \ln \bar{k})]$$

$$= \frac{1}{\sigma^2}\cdot\frac{k^\psi\bar{k}^{1-\psi}\cdot\bar{m}}{(k^\psi\bar{k}^{1-\psi} + \bar{m})^2}\cdot\ln\left(\frac{k}{\bar{k}}\right)$$

$$= \frac{1}{\sigma^2}\cdot(1-\pi)\cdot\pi\cdot\ln\left(\frac{k}{\bar{k}}\right).$$

This is equation (10) in our text.

Making use of (B1) and (A6), we can derive from (A7):

$$\text{(B2)}\quad \frac{\partial f_\sigma(k)}{\partial \sigma} = \frac{1}{\sigma^2}\cdot\frac{\partial f_\sigma(k)}{\partial \psi}$$

$$= \frac{1}{\sigma^2}\cdot \bar{y}\cdot\frac{k}{\bar{k}}\cdot\left[\left(\frac{\bar{\pi}}{\pi}\right)^{1/\psi}\cdot\left(-\frac{1}{\psi^2}\right)\ln\left(\frac{\bar{\pi}}{\pi}\right) + \bar{\pi}^{1/\psi}\left(-\frac{1}{\psi}\right)\pi^{-1/\psi-1}\cdot\frac{\partial \pi}{\partial \sigma}\right]$$

$$= -\frac{1}{\sigma^2}\cdot\frac{1}{\psi^2}\cdot\bar{y}\,\frac{k}{\bar{k}}\left(\frac{\bar{\pi}}{\pi}\right)^{1/\psi} \times\left[\ln\left(\frac{\bar{\pi}}{\pi}\right) + (1-\pi)\ln\left(\frac{k}{\bar{k}}\right)^{\psi}\right]$$

$$= -\frac{1}{\sigma^2}\cdot\frac{1}{\psi^2}\cdot y \times\left\{\ln\left(\frac{\bar{\pi}}{\pi}\right) + (1-\pi)\ln\left(\frac{\pi}{\bar{\pi}}\cdot\frac{(1-\bar{\pi})}{(1-\pi)}\right)\right\}$$

$$= -\frac{1}{\sigma^2}\cdot\frac{1}{\psi^2}\cdot y \times\left\{\pi\ln\left(\frac{\bar{\pi}}{\pi}\right) + (1-\pi)\ln\left(\frac{1-\bar{\pi}}{1-\pi}\right)\right\},$$

which is equation (11) in the text.

Appendix C: Elasticity of Substitution and the Steady State

The equation of motion (9) implies a steady-state capital intensity k^* of:

$$\text{(C1)}\qquad k^* = \left[\frac{1-a}{\left(\frac{n}{sA}\right)^{\psi} - a}\right]^{1/\psi}.$$

Calculating the steady-state profit share π^* (for $k = k^*$) yields:

$$\text{(C2)}\qquad \pi^* = a\left[\frac{Ak^*}{f_\sigma(k^*)}\right]^{\psi} = a\left(\frac{sA}{n}\right)^{\psi} = \frac{\bar{k}^{1-\psi}}{\bar{k}+\bar{m}}\cdot\left(\frac{s\bar{y}}{n}\right)^{\psi}.$$

Using (C2) as well as (A4) and (A6) we can rewrite the steady-state capital intensity in (C1) as:

$$\text{(C3)}\quad k^* = \left[\frac{(1-a)}{a}\cdot\frac{\pi^*}{(1-\pi^*)}\right]^{1/\psi} = \left[\frac{(1-\pi)}{\pi}\cdot\frac{\pi^*}{(1-\pi^*)}\right]^{1/\psi}\cdot k = \left[\frac{(1-\bar{\pi})}{\bar{\pi}}\cdot\frac{\pi^*}{(1-\pi^*)}\right]^{1/\psi}\cdot\bar{k}.$$

This is equation (17) in our text.

With the help of (A5) we can derive from (C2):

$$\text{(C4)}\quad \frac{\partial \pi^*}{\partial \sigma} = \frac{1}{\sigma^2}\cdot\pi^*\left[\ln\left(\frac{s\bar{y}}{n}\right) - \ln\bar{k}\right] = \frac{1}{\psi}\cdot\frac{1}{\sigma^2}\cdot\pi^*\cdot\ln\left(\frac{s\bar{y}}{n\bar{k}}\right)^{\psi} = \frac{1}{\psi}\cdot\frac{1}{\sigma^2}\cdot\pi^*\cdot\ln\left[\frac{\pi^*}{a}\left(\frac{\bar{y}}{A\bar{k}}\right)^{\psi}\right] = \frac{1}{\psi}\cdot\frac{1}{\sigma^2}\cdot\pi^*\cdot\ln\left(\frac{\pi^*}{\bar{\pi}}\right).$$

This corresponds to equation (18) in the text.

Finally, we make use of (C4) in order to derive from (C3):

$$\text{(C5)}\quad \frac{\partial k^*}{\partial \sigma} = -\frac{1}{\sigma^2}\cdot\frac{1}{\psi^2}\cdot\bar{k}\cdot\left[\frac{(1-\bar{\pi})}{\bar{\pi}}\cdot\frac{\pi^*}{(1-\pi^*)}\right]^{1/\psi}$$