Distribution of Wealth and Incomplete Markets: Theory and Empirical Evidence

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Abstract

This paper analyzes the equilibrium distribution of wealth in an economy where firms' productivities are subject to idiosyncratic shocks, returns on factors are determined in competitive markets, dynasties have linear consumption functions and government imposes taxes on capital and labour incomes and equally redistributes the collected resources to dynasties. The equilibrium distribution of wealth is explicitly calculated and its shape crucially depends on market incompleteness. With incomplete markets it follows a Paretian law in the top tail and the Pareto exponent depends on the saving rate, on the net return on capital, on the growth rate of population and on portfolio diversification. On the contrary, the characteristics of the labour market mostly affects the bottom tail of the distribution of wealth. The analysis also suggests a positive relationship between growth and wealth inequality. The theoretical predictions find a corroboration in the empirical evidence of Italy and United States in the period 1987-2004.

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1 Introduction

There have been several attempts, in the economic literature, to explain the statistical regularities of the distribution of wealth first showed by Pareto (1897) (see the pioneering works by Sargan (1957), Wold and Whittle (1957) and for a review Atkinson and Harrison (1978) and Davies and Shorrocks (1999)). However, as remarked in Davies and Shorrocks (1999), "research has shifted away from a concern with the overall distributional characteristics, focusing instead on the causes of individual differences in wealth holdings". One of the main reasons of the loss of interest in this field is the lack of a precise economic interpretation of the stochastic processes driving the individual wealth. In the words of Davies and Shorrocks (1999) "[these] models lack of an explicit behavioural foundation for the parameter values and are perhaps best viewed as reduced forms". This makes the stochastic models of the distribution of wealth useless both to understand the causes of an increase in income/wealth inequality and to provide some guide to public policy. Vaughan (1978) and Laitner (1979), and, more recently, Wang (2007) and Benhabib and Bisin (2007), represent an attempt to overcome this critique.

The present paper discusses a model where the aggregate behaviour of wealth is the result of market interactions among a large number of dynasties and firms subject to idiosyncratic shocks. While taking the complexity of market interaction fully into account, our model takes a behavioral perspective on consumers, by assuming a simple linear consumption function. This allows us to explicitly calculate the equilibrium distribution of wealth when returns on factors are determined in competitive markets and government taxes capital and labour incomes and redistributes the revenues to dynasties equally. As suggested in Aiyagari (1994), the shape of the equilibrium distribution crucially depends on market incompleteness. In particular, with capital markets without any frictions and transaction costs but labour income subject to uninsurable shocks the equilibrium distribution of wealth is Gaussian, a result at odds with empirical evidence (see, e.g., Klass et al. (2006)). On the contrary when frictions and transaction costs impede full diversification of dynasties' portfolios, the shape of the top tail of the distribution follows a Paretian law. The Pareto exponent, which represents an (inverse) index of the degree of inequality of the top tail of the distribution, is computed explicitly, thus allowing us to draw conclusions on the effects which different parameters have on wealth inequality. We find, for example, that an increase in the taxation of capital income or in the diversification of dynasties' portfolios increases the Pareto exponent, whereas the impact of the saving rate or the growth rate of the population crucially depends on technology.

The bottom tail of the equilibrium distribution of wealth is instead crucially affected by the characteristics of labour market and, in particular, by the cross-section distribution of wages. With a labour market completely decentralized, so that individual wages immediately respond to idiosyncratic shocks to firms, the support of the equilibrium distribution of wealth includes negative values; on the contrary if all workers receive the same wage equal to the expected marginal return of labour, i.e. bargaining in the labour market is completely centralized and workers do not bear any risk, shocks are only transmitted through return on capital and the distribution of wealth is bounded away from zero.

Finally, we show that, if the growth rate of the economy is endogenous, there is a negative relationship between the latter and the Pareto exponent, i.e. wealth inequality.

In the final section we compare our theoretical results with the empirical evidence. We study the recent trends of wealth inequality in Italy (1987-2004) and United States (1989-2004); the analysis is respectively based on the Survey of Household Income and Wealth (SHIW) and on the Survey of Consumer Finances (SCF). In both countries the top tail of the distribution of wealth follows a Pareto distribution, whose estimated Pareto exponent is decreasing in the considered period. The theoretical picture offered by the model suggests that the factors at the root of this decline are *i)* a decrease in the taxation of capital income in both countries, and *ii)* a change in technology in favour of return on capital, for Italy, or a decrease in the saving rate, for the U.S.. The demographic factor does not seem to play a relevant role in both countries. Finally, we argue that the increase in the size of the bottom tail of the distribution of wealth in both countries (more evident in Italy) can be explained by the increase in the cross-section variance of their distribution of labour incomes.

The theoretical model is in the spirit of Vaughan (1978), but he considers only two classes of individuals and returns on factors are not determined in competitive markets. Shorrocks (1975) proposes a similar approach, but he does not consider a general competitive equilibrium. The same remark applies to Pestieau and Possen (1979) and Champernowne and Cowell (1998). Garcia-Peñalosa and Turnovsky (2005) proposes a model similar to ours, but they assume an AK technology and, overall, aggregate shocks to production. Wang (2007) and Benhabib and Bisin (2007) present models close to ours, where the equilibrium distribution of wealth shows fat tails, but both consider exogenous returns on factors. In particular both Wang (2007) and Benhabib and Bisin (2007) assume that the return on wealth is constant and that labour income follows an exogenous stochastic process (Wang (2007)) or is zero (Benhabib and Bisin (2007)). Finally, Levy (2003) discusses the properties of the stochastic process governing the individual accumulation of wealth in order to have a Pareto law. He finds that all agents must have the same investment talent. In our model this happens because capital market are competitive: this ensures that every dynasty has the same investment opportunities. In this respect we generalize Levy (2003)'s results considering also labour incomes and calculating the analytical expression of the equilibrium distribution of wealth. Chatterjee et al. (2005) review several studies of the distribution of wealth with an emphasis on empirical distribution and on simple mechanistic stochastic processes which can reproduce them. A general conclusion is that the Pareto distribution arises from the combination of a multiplicative accumulation process and an additive term, as in Kesten processes (see Cont and Sornette (1997)). The same mathematical structure emerges in our model, where returns on financial investment enter in a multiplicative manner whereas labour market affects the dynamics additively. Finally, Atkinson (2007) provides a detailed discussion on the dynamics of the distribution of wealth in some OECD countries and of their possible theoretical explanations; his findings largely agree with the results of our theoretical model and with our empirical evidence. Piketty (2006) et al. provide further evidence on the factors affecting the dynamics of wealth inequality in France.

The paper is organized as follows: Section 2 presents the theoretical model; Section 3 shows the evolution of wealth distribution and characterizes the properties of the equilibrium distribution of wealth. Section 4 discusses the empirical evidence supporting our theoretical results. Section 5 concludes. All proofs are relegated in the appendix.

2 The Model

We model a competitive economy in which firms demand capital and labour. We assume all the wealth is owned by dynasties, who inelastically offer capital and labour and decide which amount of their disposable income is saved. Wages and returns on capital adjust to clear the labour and capital markets respectively. For the sake of simplicity we consider just one type of capital and no risk-free asset.[1] Hence human

[1]The inclusion of a risk-free asset does not modify in any substantial way the properties of the economy.

capital can be represented by different labour endowments and/or included in the capital stock (in the latter case it is accumulated at the same rate of physical capital).

From a technical point of view, we follow a standard approach to model a stochastic economy, see, e.g., Chang (1988) and Garcia-Peñalosa and Turnovsky (2005). In particular, we derive continuum time stochastic equations for the evolution of the distribution of wealth first specifying the dynamics over a time interval $[t, t+dt)$ and then letting $dt \to 0$.

2.1 Firms

Consider an economy with F firms. Every firm j has the same technology $q(\cdot)$. Its output over the period $[t+dt)$, $dy_j(t)$, is the joint product of its technology and of a random idiosyncratic component $dA_j(t)$:

$$dy_j(t) = q[k_j(t), l_j(t)]dA_j(t), \tag{1}$$

where $k_j(t)$ and $l_j(t)$ are respectively the capital and the labour of firm j at time t and dA_j is a random shock to production. We assume that at time t firm j knows only the distribution of dA_j (see Section 2.4 for the characteristics of the stochastic components of the economy).

The presence of a labour augmenting exogenous technological progress can be taken into account assuming that $l_j(t) = l_j^*(t)\exp(\psi t)$, where ψ is the exogenous growth rate of technological progress. Here we shall confine our discussion to the $\psi = 0$ case. Indeed, all the following analysis remains the same, except for the meaning of the per capita variables, which are to be interpreted in efficient units of labour (see Chang (1988), p. 163).

We make the standard assumption that $q(\cdot)$ is an homogeneous function of degree one (i.e. technology has constant returns to scale), with positive first derivatives and negative second derivatives with respect to both arguments. Hence:

$$q(k, l) = lg(\lambda) \text{ with } g'(\lambda) > 0 \text{ and } g''(\lambda) < 0, \tag{2}$$

where $q(k/l, 1) = g(\lambda)$ and $\lambda = k/l$ is the capital per worker.

Every firm j maximizes its expected profits $d\pi_j$ over the period $[t, t+dt)$:

$$\max_{k_j(t), l_j(t)} E[d\pi_j(t)] = \max_{k_j(t), l_j(t)} q[k_j(t), l_j(t)]E[dA_j(t)] - k_j(t)dr(t) - l_j(t)dw(t), \tag{3}$$

at given $dr(t)$ and $dw(t)$, the expected return on capital and the expected wage respectively over the period $[t, t+dt)$. From the first order conditions of Problem (3) we have that:

$$dr\left(t\right)=\frac{\partial q}{\partial k_j\left(t\right)}E\left[dA_j\left(t\right)\right]\text{ and} \tag{4}$$

$$dw\left(t\right)=\frac{\partial q}{\partial l_j\left(t\right)}E\left[dA_j\left(t\right)\right]. \tag{5}$$

Since $q\left(\cdot\right)$ is an homogeneous function of degree one we have:

$$q[k_j\left(t\right),l_j\left(t\right)]=\frac{\partial q}{\partial k_j\left(t\right)}k_j\left(t\right)+\frac{\partial q}{\partial l_j\left(t\right)}l_j\left(t\right), \tag{6}$$

which with Eqq. (3), (4) and (5) implies that the expected profits of each firm are zero.

After the realization of shock dA_j firm j gets its output and it rewards its factors according to their marginal productivity (see Eqq. (4) and (5)):

$$dr_j\left(t\right)=\frac{\partial q}{\partial k_j\left(t\right)}dA_j\left(t\right)=\frac{dr\left(t\right)dA_j\left(t\right)}{E\left[dA_j\left(t\right)\right]}\text{ and} \tag{7}$$

$$dw_j\left(t\right)=\frac{\partial q}{\partial l_j\left(t\right)}dA_j\left(t\right)=\frac{dw\left(t\right)\ dA_j\left(t\right)}{E\left[dA_j\left(t\right)\right]}. \tag{8}$$

Under Eqq. (7) and (8) the realized profits are zero and both capital and labour are bearing risk. This arrangement for the returns on capital reflects, for example, the case where all lenders of capital are shareholders (i.e. no risk-free bonds are available) and profits are entirely distributed.

For the return on labour Eq. (8) assumes state-contingent wages. This captures the observed volatility in earnings (see, e.g. Deaton (1991)), but it contrasts with the general wisdom that wages are set independent of the realized state of Nature because workers are risk-adverse. The implications of staggered wages, i.e. when workers bear no risk, will be considered in Section 3.2.3.

2.2 Dynasties

The economy is populated by N dynasties. We use a subscript i to denote dynasty i. Let l_i and p_i be respectively the average endowment of labour and the current average wealth of member of dynasty i.[2] Her gross income, dy_i, over the period $[t,t+dt)$ is given by:

$$dy_i\left(t\right)=p_i\left(t\right)\sum_{j=1}^{F}\theta_{i,j}\left(t\right)dr_j\left(t\right)+l_i\sum_{j=1}^{F}\phi_{i,j}\left(t\right)dw_j\left(t\right), \tag{9}$$

[2] Different amounts of labour could be interpreted as different level of abilities (productivities) among individuals. In fact, the wage rate is defined in terms of an unit of labour service: the labour income for dynasty i is equal to $l_i dw$.

where $\theta_{i,j}(t)$ is the fraction of wealth p_i invested in firm j at time t ($\sum_{j=1}^{F}\theta_{i,j} = 1$) and $\phi_{i,j}(t)$ is the fraction of labour that dynasty i employs in firm j at time t ($\sum_{j=1}^{F}\phi_{i,j} = 1$). Coefficients $\phi_{i,j}(t)$ and $\theta_{i,j}(t)$ should be thought of as the resulting allocation arising from market interactions at time t. For given vectors of returns on capital and wages, $\phi_{i,j}(t)$ and $\theta_{i,j}(t)$ depend on the individual preferences. In particular, risk-adverse dynasties would prefer a maximally diversified portfolio, i.e. $\theta_{i,j}(t) = 1/F$, and a labour endowment maximally spread across all firms, i.e. $\phi_{i,j}(t) = 1/F$ (their expected income is indeed independent of portfolio and labour allocation). On the other hand possible transaction costs and frictions in the labour and capital markets, i.e. market incompleteness, may result in a more concentrated allocation.[3] Actually, the degree of concentration, as a reflection of market incompleteness, will play a major role on the equilibrium distribution of wealth.[4]

The disposable income is the result of taxation and redistribution. We assume that capital and labour income are taxed at a flat rate τ_k and τ_l respectively. The resources collected from taxes are redistributed to dynasties by lump-sum transfers. Therefore the dynasty i's disposable income dy_i^D over the period $[t, t+dt)$ is given by:[5]

$$dy_i^D = (1-\tau_k)\, p_i \sum_{j=1}^{F}\theta_{i,j} dr_j + (1-\tau_l)\, l_i \sum_{j=1}^{F}\phi_{i,j} dw_j + \frac{\tau_k}{N}\sum_{j=1}^{F} k_j dr_j + \frac{\tau_l}{N}\sum_{j=1}^{F} l_j dw_j \qquad (10)$$

We assume that dynasty i consumes according to the following Keynesian consumption function:[6]

$$dc_i = d\bar{c} + c\,dy_i^D + dc^p p_i, \qquad (11)$$

where $c \in (0,1)$ is the marginal propensity to consume with respect to disposable income, $d\bar{c}$ captures the presence of a minimum consumption $d\bar{c} \geq 0$, which is independent of the level of income and dc^p is the marginal propensity to consume with respect to wealth.

The choice of a simple linear "reduced form" consumption function for dynasties in Eq. (11) reflects the main focus of the paper, which is not on optimizing behaviour under uncertainty,[7] but rather on explaining the aggregate behaviour of wealth as the result of market interactions among a large number of dynasties and firms subject to

[3]For example, the cost of information in capital market can be seen as a fixed cost which may limit market participation, see Arrow (1996).

[4] Guiso et al. (2001) report ample empirical evidence that actual household portfolios do not conform with the theoretical prediction of optimizing risk-adverse dynasties in complete markets.

[5]In the following we will omit the time index of the variables if this is not source of confusion.

[6]The consumption function in the original Keynesian version does not include a term for wealth.

[7]The derivation of consumption function from an inter-temporal optimization problem under un-

idiosyncratic risk. Eq. (11) represents therefore a first cut which includes the key ingredients of consumption behaviour, as disposable income and wealth, in a simple linear fashion.[8] Most importantly, the cost of using a behavioural rule for consumption is more than outweighed by the understanding we shall derive on the elements (e.g. parameters of production, the saving or tax rates) that drive the wealth distribution.[9] Finally, the linear consumption function finds an empirical corroboration in the Italian data, as shown in Section 4.

Given Eq. (11), dynasty i accumulates her wealth according to:

$$dp_i = sdy_i^D - d\bar{c} - dn_i p_i, \tag{12}$$

where the first term, sdy_i^D ($s \equiv 1 - c$), reflects the relationship between savings and disposable income ($s \in (0, 1)$ is the marginal saving rate). The last term, $dn_i p_i$, arises

certainty lacks of a closed solution, excluding a very limited number of cases. This would prevent us from finding an analytical expression of the equilibrium distribution of wealth. Moreover, Chang (1988) shows that Eq. (11) with $d\bar{c} = dc^p = 0$ can represent the consumption function of an intertemporal optimizing agent with infinite lifetime when utility function is CES and technology is Cobb-Douglas; in such case Chang (1988), p. 163, shows that the saving rate $(1 - c)$ is equal to the intertemporal elasticity of substitution. Wang (2007) and Benhabib and Bisin (2007) represent other examples where the agent's optimizing behaviour leads to a linear consumption function in wealth and labour income like in Eq. (11). Cagetti and DeNardi (2005) survey the models of the distribution of wealth with intertemporal optimizing individuals. The key insight of this approach is that saving rate is decreasing with income/wealth. This is in contrast with the empirical evidence discussed in Section 4; moreover, this does not allow to explain the Pareto-like shape of the upper tail of the distribution of wealth (see Cagetti and DeNardi (2005)). This is confirmed by an extension of our model where dc^p is made an increasing function of p_i, i.e. saving rate is decreasing with p_i. In an intertermporal optimization framework under uncertainty with many assets, further problematic issues of consistency with empirical evidence arise, such as the *equity premium puzzle* (see, e.g., Romer (2005), Cap. 7) and inertia in asset allocation (see, e.g., Brunnermeier and Nagel (2008)).

[8]From another point of view Eq. (11) can be thought of as the leading order expansion of a generic consumption function with arguments disposable income and wealth, which neglects higher order differentials.

[9]A limit of our approach is that it ignores the possible effects on saving rates, due to changes in the fiscal policy. However, ex-ante such effects are ambiguous. In the empirical section we show that, in response to a decline in the tax rates both in Italy and in U.S., the saving rate has increased in Italy and decreased in U.S.. In Italy such increase could be partially explained by precautionary saving, caused by the increase in the expected volatility of future incomes (see Jappelli and Pistaferri (2000b)) and by the change in social security in 1990s (see Attanasio and Brugiavini (2003)). In this regard, for our purposes it is preferable to take saving rates as a behavioural parameter to be estimated, being the latter observable, rather than to consider other variables, such as the expectations of future incomes, and/or changes in welfare policy, which are not easily measurable.

from the term dc^p, but it also includes demographic effects; if $d\tilde{n}_i$ represents the number of newborns over the period $[t, t+dt)$ in dynasty i, then $dn_i = d\tilde{n}_i + dc^p$. Therefore we are implicitly assuming that every member of dynasty i is endowed with the same level of wealth. The term $dn_i p_i$ may also include any other effect which can directly affect dynasty i's wealth, as a possible real estate tax (see Benhabib and Bisin (2007)).[10] For the sake of simplicity, in the following we will refer to dn_i as a *demographic* component.

2.3 Equilibrium

In the equilibrium of capital markets we have that:

$$P = \sum_{i=1}^{N} p_i = \sum_{j=1}^{F} k_j = K, \tag{13}$$

while in the equilibrium of labour market:

$$\sum_{i=1}^{N} l_i = \sum_{j=1}^{F} l_j = L. \tag{14}$$

From Eqq. (7) and (8) we have:

$$E\left[dr_j\right] = dr \;\forall j \text{ and} \tag{15}$$

$$E\left[dw_j\right] = dw \;\forall j. \tag{16}$$

From Eqq. (4), (5), (15) and (16) we have that:

$$\frac{k_j}{l_j} = \frac{K}{L} = \frac{P}{L} = \lambda \;\forall j, \tag{17}$$

that is every firm j uses the same production technique λ; the latter is also the firms' endowment of capital per unit of labour. For convenience we also define the per capita wealth $\bar{p} = \Sigma_{i=1}^{N} p_i / N = P/N$ and the per capita labour endowment $\bar{l} = \Sigma_{i=1}^{N} l_i / N = L/N$. In equilibrium firm j rewards its factors at the following rates:

$$dr_j = dA_j g'(\lambda) \text{ and} \tag{18}$$

$$dw_j = dA_j \left[g(\lambda) - \lambda g'(\lambda)\right]. \tag{19}$$

The allocation of capital and labour among firms should be such to satisfy the equilibrium conditions (13)-(17). For example, the case in which each dynasty i works just in

[10] We thank Prof. Shyan Sunder for pointing us such possibility.

a single firm $j^*(i)$ (i.e. $\phi_{i,j^*(i)} = 1$ and $\phi_{i,j} = 0$ for $j \neq j^*(i)$) *may not* be compatible with market equilibrium because this may not ensure that every firm has the optimal ratio of capital and labour λ (see Eq. (17)). On the other hand the optimal allocation in complete markets, where each (risk-adverse) dynasty invests an equal share of her wealth in each firm (i.e. $\theta_{i,j} = 1/F$) and contributes the same amount of labour to each firm (i.e. $\phi_{i,j} = 1/F$), is always compatible with market equilibrium. With incomplete markets (i.e. if frictions and transaction costs prevents dynasties from investing and/or working in all firms), consistency with market equilibrium implies that the coefficients $\theta_{i,j}$ and $\phi_{i,j}$ carry some dependence on the dynamics of p_i. In this situation, market equilibrium is not sufficient to determine unambiguously the value of coefficients $\theta_{i,j}$ and $\phi_{i,j}$. In what follows, we will not explicitly model frictions and transaction costs. Rather, we shall treat $\theta_{i,j}$ and $\phi_{i,j}$ as parameters of the economy specifying the degree of concentration of capital investment and labour allocation among firms induced by frictions and transaction costs. We find it convenient to introduce the variables

$$\Theta_{i,i'} = \sum_{j=1}^{F} \theta_{i,j}\theta_{i',j}, \quad \Omega_{i,i'} = \sum_{j=1}^{F} \theta_{i,j}\phi_{i',j} \text{ and } \Phi_{i,i'} = \sum_{j=1}^{F} \phi_{i,j}\phi_{i',j}. \tag{20}$$

These characterize the degree of intertwinement of economic interactions, i.e. how random shocks *propagate* throughout the economy. For example $\Theta_{i,i'}$ is a scalar which represents the overlap of investments of dynasty i with those of dynasty i'.

2.4 The Continuum Time Limit

Let us now make the dependence of differentials on the time infinitesimal dt explicit. In particular, we take:

$$dr = \rho dt; \tag{21}$$

$$dw = \omega dt; \tag{22}$$

$$dn = \nu dt; \tag{23}$$

$$d\bar{c} = \chi dt \text{ and} \tag{24}$$

$$E\left[dA_j\right] = adt \; \forall j, \tag{25}$$

where ρ is the interest rate, ω the wage rate, $a > 0$ a scale parameter, $\nu > 0$ the growth rate of population plus the marginal effect of wealth on consumption and $\chi \geq 0$ the minimum consumption.

For productivity shocks we define:

$$d\zeta_j = \frac{dA_j - E\left[dA_j\right]}{a} \tag{26}$$

so that $E[d\zeta_j] = 0$. We assume that random shocks $d\zeta_j$ are independent across both firms and time, i.e. $d\zeta_j$ are independent Wiener increments with the following properties:

$$E[d\zeta_j d\zeta_{j'}] = \Delta dt \delta_{j,j'} \delta(t - t') \text{ and } \tag{27}$$

where Δ is the variances of productivity shocks, $\delta_{j,j'} = 1$ if $j = j\prime$ and $\delta_{j,j'} = 0$ otherwise, whereas $\delta(t - t')$ is Dirac's δ distribution.

Finally, from Eqq. (7)-(8), (21)-(25) and (26) we have:

$$\begin{aligned} dr_j &= \rho dt + \rho d\zeta_j \text{ and} \\ dw_j &= \omega dt + \omega d\zeta_j. \end{aligned} \tag{28}$$

Given these definitions the dynamics of dynasty i's wealth is given in Proposition 1.

Proposition 1 *The dynasty i's wealth obeys the following stochastic differential equation:*[11]

$$\frac{dp_i}{dt} = s\left[(1 - \tau_k)\rho p_i + (1 - \tau_l)\omega l_i + \tau_k \rho \bar{p} + \tau_l \omega \bar{l}\right] - \chi - \nu p_i + \eta_i, \tag{29}$$

where η_i is a white noise term with $E[\eta_i(t)] = 0$ and covariance:

$$E[\eta_i(t)\eta_{i'}(t')] = \delta(t - t') H_{i,i'}[\vec{p}], \tag{30}$$

where

$$\begin{aligned} H_{i,i'}[\vec{p}] = \Delta s^2 \Big\{ & (1-\tau_k)^2 \rho^2 p_i p_{i'} \Theta_{i,i'} + (1-\tau_l)^2 \omega^2 l_i l_{i'} \Phi_{i,i'} + \\ & + (1-\tau_k)(1-\tau_l)\rho\omega \left[p_i l_{i'} \Omega_{i,i'} + l_i p_{i'} \Omega_{i',i}\right] + \\ & + \frac{\tau_k \rho + \tau_l \omega/\lambda}{N} \left[(1-\tau_k)\rho(p_i \vartheta_i + p_{i'} \vartheta_{i'}) + (1-\tau_l)\omega(l_i \varphi_i + l_{i'} \varphi_{i'})\right] + \\ & + \frac{\left[\tau_k \rho + \tau_l \omega/\lambda\right]^2}{N^2} \sum_{j=1}^{F} k_j^2 \Big\}, \end{aligned}$$

and

$$\vartheta_i = \sum_{i'=1}^{N} \Theta_{i,i'} p_{i'}, \quad \varphi_i = \sum_{i'=1}^{N} \Omega_{i,i'} p_{i'}. \tag{31}$$

Proof. See Appendix A. ■

Eq. (30) shows that the correlation in the shocks hitting two dynasties i and i' arises either because both are investing in the same firms ($\Theta_{i,i'}$), or because one is investing in the firm in which the other is working ($\Omega_{i,i'}$) or because both are working in the same firm ($\Phi_{i,i'}$). Terms ϑ_i and φ_i are respectively the average capital of the firms where dynasty i is investing and working.

[11] Here we adopt the notation of Langevin equations, see Gardiner (1997), p. 80.

3 Infinite Economy

In this section we analyze the properties of the infinite economy, that is of an economy where N and $F \to \infty$. In particular, we assume that $F = fN$, where f is a positive constant. This assumption is not a relevant limitation of the analysis because in a real economy N and F may be of the order of some millions (in general F represents the number of possible different types of investment).

For the sake of simplicity we assume that dynasties do not differ among themselves in their endowment of labour l_i, in the diversification of their portfolios $\Theta_{i,i}$, in the allocation of their wealth among the firms where they are working $\Omega_{i,i}$ and in the number of firms where they are working $\Phi_{i,i}$, i.e. we assume that:

$$l_i = \bar{l} = 1 \quad \forall i; \tag{32}$$

$$\Theta_{i,i} = \bar{\Theta} \quad \forall i; \tag{33}$$

$$\Omega_{i,i} = \bar{\Omega} \quad \forall i \text{ and} \tag{34}$$

$$\Phi_{i,i} = \bar{\Phi} \quad \forall i. \tag{35}$$

Notice that $\Theta = 1$ means no diversification of the dynasties' portfolios, whereas $\Theta = 1/F$ (i.e. $\Theta \to 0$ for $F \to \infty$) corresponds to maximal diversification of portfolios; similarly, $\Phi = 1$ means that each dynasty is working in just one firm (the opposite implausible case is $\Phi = 0$, when each dynasty works in all firms).

We focus on the properties of equilibrium, i.e. on the behaviour of the infinite economy in the limit $t \to \infty$. We first derive the evolution of the aggregate variables and we then focus on two types of economies: i) the stationary/exogenous growth economy and ii) the endogenous growth economy.

3.1 Evolution of Aggregate Variables

In order to derive the dynamics of per capita wealth $\bar{p}$ from Eq. (29) we have:

$$\frac{1}{N}\frac{d}{dt}\sum_{i=1}^{N} p_i = \frac{d\bar{p}}{dt} = s\left(\rho\bar{p} + \omega\right) - v\bar{p} - \chi + \bar{\eta}, \tag{36}$$

where

$$\bar{\eta}dt = \frac{1}{N}\sum_{i=1}^{N}\left[dp_i - E\left(dp_i\right)\right] = \frac{s(\rho + \omega/\lambda)}{N}\sum_{j=1}^{F} k_j d\zeta_j.$$

Proposition 2 shows under which conditions the term $\bar{\eta}dt$ in Eq. (36) can be neglected, so that in the limit of infinite economy the dynamics of $\bar{p}$ is deterministic.[12]

Proposition 2 *Assume that there exists a constant $\bar{\theta} > 0$ such that $\forall t$:*

$$\sum_{i=1}^{N} \theta_{i,j} \leq \bar{\theta} \; \forall j, N \tag{37}$$

and that:

$$\lim_{N\to\infty} \sum_{i=1}^{N} \left(\frac{p_i}{P}\right)^2 = 0. \tag{38}$$

Then $\forall \epsilon > 0$

$$\lim_{N,F\to\infty} P\left(\left|d\bar{p} - \left[s\left(\rho\bar{p} + \omega\right) - v\bar{p} - \chi\right] dt\right| > \epsilon\right) = 0$$

and, in the limit $N, F \to \infty$, the per capita wealth $\bar{p}$ almost surely follows a deterministic dynamics given by:

$$\frac{d\bar{p}}{dt} = s\left(\rho\bar{p} + \omega\right) - \chi - v\bar{p}. \tag{39}$$

Proof. See Appendix B. ■

Assumption (37) states that the total share of dynasties' investment in each firm is bounded above, that is investments across firms cannot be too concentrated.[13] Taking the sum on j in Assumption (37) yields $\bar{\theta} \geq 1/f = N/F$, i.e. the number of firms must grow at least as fast as the number of dynasties.

Assumption (38) states that as the number of dynasties goes to infinity the share of wealth of every dynasty on the total wealth of the economy must converge to zero. More technically, Assumption (38) is a law of large numbers and, in particular, it is an assumption on the shape of the top tail of the distribution of p_i. In fact, if the probability distribution density of p_i behaves for large p as $f(p) \sim p^{-\alpha-1}$ with $\alpha > 1$, as we will find later in Propositions 4 and 12, then Assumption (38) holds. If Assumption (38) does

[12]With idiosyncratic shocks to firms the occurrence of skewed distributions of firm sizes has been suggested to be an important source of fluctuations in aggregate variables (see Gabaix (2008)). In our model it can be shown that under the assumption that every dynasty i allocates an equal share of its wealth to a limited number of firms, i.e. $\theta_{i,j} = \Theta$ in $1/\Theta$ firms and $\theta_{i,j} = 0$ in $F - 1/\Theta$ firms, and the distribution of wealth has a power law behavior in the upper tail (as it will happen in our economy), the firm size distribution has a power law behaviour in the upper tail with the same Pareto exponent of wealth distribution. However, Assumption 38 rules out aggregate fluctuations. A detailed analysis of the firm size distribution and of economies with aggregate fluctuations goes beyond the scope of the present paper and it is doomed to future research.

[13]As extreme example consider the case where all dynasties invest in the same firm $j = 1$ all their capital, i.e. $\theta_{i,1} = 1$ and $\theta_{i,j} = 0$ for $j > 1 \; \forall i$. Then Condition (37) is violated for $j = 1$.

not hold then per capita wealth follows a stochastic process and the dynasty i's wealth will fluctuate both because of the idiosyncratic shocks and because of the fluctuations of the aggregate variables.

Substituting for ρ and ω in Eq. (39) from Eqq. (15)-(16) and (21)-(25) we get:

$$\frac{d\bar{p}}{dt} = sag\left(\bar{p}\right) - \chi - v\bar{p}, \tag{40}$$

which is the well-known equation of the Solow growth model (without capital depreciation) and augmented with the minimum consumption χ.[14] Stiglitz (1969) shows that Eq. (40) can generate many different dynamics according to the parameters' value and to the shape of production function $g(\cdot)$. In particular, the per capita wealth of economy $\bar{p}$ could i) converge to a positive value/growing at an exogenous rate, ii) grow at an endogenous rate or, finally, iii) converge towards zero; the long-run equilibrium could also depend on the initial level of the per capita wealth. To our purposes economy iii) is trivial, so that in the following we analyse only economies i) and ii).

3.2 Stationary/Exogenous Growth Economy

Heuristically the condition to have an equilibrium with constant per capita wealth is that the growth rate of per capita wealth becomes negative for large value of $\bar{p}$. Moreover, depending on the value of the production function in zero, we can have zero, one or two equilibria (but at most one will be stable). With an exogenous technological progress at a rate ψ, $\bar{p}$ is the per capita wealth measured in efficient units; therefore in equilibrium the per capita wealth will grow at the exogenous growth rate of technological progress ψ.

Proposition 3 states the conditions for the existence of an equilibrium with a constant and positive per capita wealth.

Proposition 3 *Assume that $g(\cdot)$ satisfies Assumption (2), the dynamics of the per capita wealth of the economy obeys Eq. (40) and that*

$$\lim_{\bar{p}\to\infty} g'\left(\bar{p}\right) < \frac{\nu}{sa}. \tag{41}$$

Then if

$$g\left(0\right) > \frac{\chi}{sa} \tag{42}$$

[14] $ag\left(\bar{p}\right) = aq\left(\bar{k}, 1\right)$ is the per capita output adjusted for the effective supply of labour of each individual, i.e. it would be per-capita output if $L = N$

an equilibrium with constant and positive per capita wealth exists. Otherwise if:

$$g(0) < \frac{\chi}{sa} \tag{43}$$

and if

$$\exists \bar{p}_1 < \bar{p}_2 \text{ such that } sag(\bar{p}_h) = \chi + v\bar{p}_h \text{ for } h = 1, 2, \tag{44}$$

then $\bar{p}_2$ and $\bar{p}_1$ are respectively a local stable and unstable equilibrium. The economy converges towards an equilibrium with a per capita wealth equal to $\bar{p}_2$ if and only if $\bar{p}(0) > \bar{p}_1$, while if $\bar{p}(0) < \bar{p}_1$ the economy converges towards an equilibrium with zero per capita wealth.
If a stable equilibrium with positive per capita wealth exists, then the equilibrium per capita wealth, $\bar{p}^*$, solves:

$$sag(\bar{p}^*) = \chi + v\bar{p}^*, \tag{45}$$

while the interest rate and the wage rate are respectively given by:

$$\rho^* = ag'(\bar{p}^*) \text{ and } \tag{46}$$

$$\omega^* = a[g(\bar{p}^*) - \bar{p}^* g'(\bar{p}^*)] \tag{47}$$

Proof. See Appendix C. ■

Figures 1 and 2 provide the intuition of results in Proposition 3.

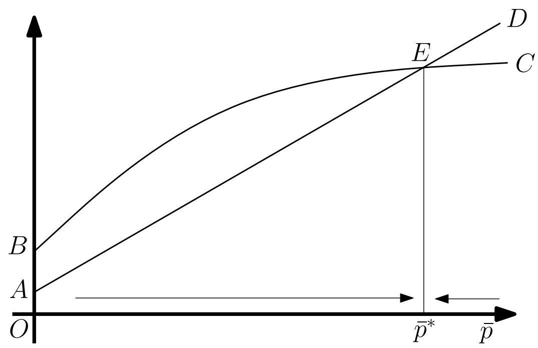

Figure 1: economy with a global stable equilibrium.

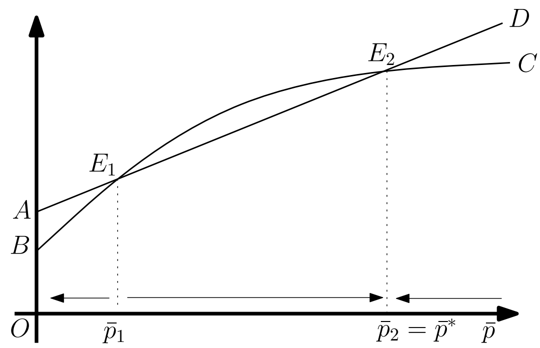

Figure 2: economy with a local stable equilibrium.

Figure 1 shows the economy with a single global stable equilibrium, whereas Figure 2 refers to the economy with two equilibria, only one of which is stable (E_2). In the latter economy, in order to have an equilibrium with a positive per capita wealth, it is necessary that the initial per capita wealth is higher than $\bar{p}_1$.

Proposition 3 shows that $\bar{p}^*$ positively depends on s and a and negatively on ν.[15] The equilibrium interest rate ρ^* negatively depends on $\bar{p}^*$; on the contrary, the effect of

[15] This is straightforward given Eq. (45) and Assumption (2) on $g(\cdot)$.

a on ρ^* is ambiguous because it has a positive direct effect, but also a negative induced effect: in fact, the increase in $\bar{p}^*$, caused by the increase in a, tends to decrease ρ^*. Finally, the equilibrium wage rate ω^* increases with $\bar{p}^*$.

3.2.1 The Equilibrium Distribution of Wealth

In the following we characterize the equilibrium distribution of wealth of dynasty i, when the economy converges toward a constant and positive per capita wealth $\bar{p}^*$; we then show that Assumption (38) is satisfied in equilibrium.

Proposition 4 *Assume that the infinite economy converges towards a positive and constant per capita wealth $\bar{p}^*$ given by Eq. (45), and that $\chi < s\left[\omega^* + \tau_k \rho^* \bar{p}\right]$. Let $f(p_i)$ be the equilibrium distribution of p_i. Then:*

- *if $\bar{\Theta}, \bar{\Omega} = \bar{\Phi} = 0$ (complete markets) then:*

$$f(p_i) = \delta(p_i - \bar{p}) . \tag{48}$$

- *if $\bar{\Theta}, \bar{\Omega} = 0$ and $\bar{\Phi} > 0$ (capital markets without frictions and transaction costs) then:*

$$f(p_i) = \mathcal{N} e^{-\frac{(z_0 - z_1 p_i)^2}{z_1 a_0}}; \tag{49}$$

- *if $\bar{\Theta}, \bar{\Omega}, \bar{\Phi} > 0$ (incomplete markets) then:*

$$f(p_i) = \left[\frac{\mathcal{N}}{\left(a_0 + a_1 p_i + a_2 p_i^2\right)^{1 + z_1/a_2}}\right] e^{4\left[\frac{z_0 + z_1 a_1/(2a_2)}{\sqrt{4a_0 a_2 - a_1^2}}\right] \arctan\left(\frac{a_1 + 2a_2 p_i}{\sqrt{4a_0 a_2 - a_1^2}}\right)}; \tag{50}$$

where

$$\begin{aligned}
z_0 &= s\left[\omega^* + \tau_k \rho^* \bar{p}\right] - \chi; \\
z_1 &= \nu - s\left(1 - \tau_k\right)\rho^*; \\
a_0 &= \Delta s^2 \left(1 - \tau_l\right)^2 \omega^{*2} \bar{\Phi}; \\
a_1 &= 2\Delta s^2 (1 - \tau_k)(1 - \tau_l)\rho^* \omega^* \bar{\Omega} \ \textit{and} \\
a_2 &= \Delta s^2 \left(1 - \tau_k\right)^2 \rho^{*2} \bar{\Theta},
\end{aligned}$$

where $\mathcal{N}$ is a constant defined by the condition $\int_{-\infty}^{\infty} f(p_i)\, dp_i = 1$.

Proof. See Appendix D. ■

Condition $s\left[\omega^* + \tau_k \rho^* \bar{p}\right] > \chi$ is not a source of limitation of the analysis: it ensures that dynasties with zero initial wealth have an *expected* positive saving, i.e. they can escape from the zero wealth trap.

When risk-adverse dynasties can fully diversify both their income from capital investment and labour (i.e. $\theta_{i,j} = \phi_{i,j} = 1/F$), in an infinite economy, they can eliminate all sources of risk, i.e. $\bar{\Theta}, \bar{\Omega} = \bar{\Phi} = 0$. Therefore their income is deterministic and, in equilibrium, they all end up with the same wealth, i.e. $p_i = \bar{p}$.

When capital markets are without any frictions and transaction costs, but each dynasty can work in a limited number of firms, risk-adverse dynasties fully diversify their portfolios ($\theta_{i,j} = 1/F$) but labour income remains subject to uninsurable idiosyncratic shocks; then in an infinite economy $\bar{\Theta}, \bar{\Omega} = 0$ and $\bar{\Phi} > 0$. As a consequence, the equilibrium distribution of wealth is affected only by shocks on wages and it attains a Gaussian shape with mean $z_0/z_1 = \bar{p}$ and variance $a_0/\left(2z_1\right)$.

In the more realistic incomplete market case, i.e. $\bar{\Theta}, \bar{\Omega}, \bar{\Phi} > 0$, for large p_i $f\left(p_i\right) \sim p_i^{-\alpha-1}$ follows a Pareto distribution whose exponent is given by:

$$\alpha = 1 + 2z_1/a_2 = 1 + 2\frac{\nu - s\left(1-\tau_k\right)\rho^*}{\Delta s^2 (1-\tau_k)^2 \rho^{*2} \bar{\Theta}}. \tag{51}$$

We observe that z_1, $a_2 > 0$ (see Condition (41) and Eq. (46)) and hence $\alpha > 1$: this ensures that Assumption (38) is indeed satisfied in equilibrium.

Assumptions (32)-(35) eliminate cross-dynasty heterogeneity in Eq. (50); hence, the latter can be directly compared with the empirical distributions of wealth discussed in Section 4. However, it is worth noting that if dynasties were heterogeneous in their portfolio diversification, i.e. $\Theta_{i,i} \neq \Theta_{i',i'}$, then the top tail distribution would be populated by the dynasties with the highest $\Theta_{i,i}$, that is by those dynasties with the less diversified portfolios ($\Theta_{i,i} = 1$ means that dynasty i has just one asset in its portfolio). This finding agrees with the empirical evidence on the low diversification of the portfolios of wealthy households discussed in Guiso et al. (2001), Cap. 10.

Remark 5 shows the dependence of α on the main variables and parameters of the model.

Remark 5 *The size of the top tail of the distribution of wealth measured by (the inverse of Pareto exponent) α is an increasing function of Δ and $\bar{\Theta}$, and a decreasing function of τ_k. Changes in s and v have, on the contrary, an ambiguous effect on the size of the top tail of distribution of wealth.*

A lower diversification of the dynasty i's portfolio, measured by $\bar{\Theta}$, tends to decrease α and therefore to increase the inequality in the top tail of distribution.[16] The opposite holds for τ^k (see Benhabib and Bisin (2007)). Labour income does not play any role, whereas the gross return on capital ρ^* does. An increase in the latter tends to increase the size of the top tail (i.e. $\partial\alpha/\partial\rho^* < 0$). In this respect the ambiguous relationships between α and s is the result of two competing effects: when s increases a *direct* effect tends to decrease α, while an *induced* effect tends to increase α; the latter is caused by a decrease in the return on capital ρ^*, in turn caused by an increase in the equilibrium per capita wealth $\bar{p}^*$. When ν increases the contrary happens. Without specifying the technology it is not possible to determine which effect prevails. These results highlight the importance to endogenize the returns to factors in order to study the effect on inequality of changes in the saving rate and in the growth rate of population (e.g., compare our results with the one in Atkinson and Harrison (1978), Cap. 3). In the following we analyse the case of CES technology.

3.2.2 CES Technology

The CES production function is given by $q(k,l) = [\varepsilon k^\gamma + (1-\varepsilon)\, l^\gamma]^{1/\gamma}$ with $\varepsilon \in (0,1)$ and $\gamma \in (-\infty, 1]$; the elasticity of substitution in production is equal to $1/(1-\gamma)$. For $\gamma \to 0$ we have the Cobb-Douglas production function $q(k,l) = k^\varepsilon l^{1-\varepsilon}$. The function $g(\lambda)$ is therefore given by $g(\lambda) = [\varepsilon\lambda^\gamma + 1 - \varepsilon]^{1/\gamma}$, while in the Cobb-Douglas case $g(\lambda) = \lambda^\varepsilon$. The CES production function satisfies Assumption (2).

Condition (41) for exogenous growth is satisfied for $\varepsilon^{1/\gamma} < v/(sa)$ when $\gamma \in (0,1]$, while it is always satisfied when $\gamma \leq 0$ given that v, s and $a > 0$. Condition (42) for one stable global equilibrium is satisfied for $(1-\varepsilon)^{1/\gamma} > \chi/(sa)$ when $\gamma \in (0,1]$, while it is never satisfied when $\gamma \leq 0$. With Cobb-Douglas technology, i.e. $\gamma = 0$, Conditions (41) and (43) for the existence of two equilibria are always satisfied, but $\bar{p}_1 = 0$.

[16] Pestieau and Possen (1979) also address the dependence of the distribution of wealth on portfolio choices, though in a rather different framework. Castaldi and Milakovic (2006) suggest that also the frequency of the changes in the composition of the wealthiest portfolios can affect the Pareto exponent.

Assume that $\chi = 0$; then in the equilibrium with positive per capita wealth:

$$\bar{p}^* = \left[\frac{1-\varepsilon}{(\nu/sa)^\gamma - \varepsilon}\right]^{1/\gamma}; \tag{52}$$

$$\rho^* = \varepsilon a^\gamma \left(\frac{\nu}{s}\right)^{1-\gamma}; \tag{53}$$

$$\omega^* = a\,(1-\varepsilon)^{1/\gamma}\left[\frac{(\nu/sa)^\gamma}{(\nu/sa)^\gamma - \varepsilon}\right]^{1/\gamma - 1} \text{ and} \tag{54}$$

$$\alpha = 1 + 2\left[\frac{1-(1-\tau_k)\,\varepsilon\,(as/v)^\gamma}{\Delta(1-\tau_k)^2\varepsilon^2\,(as)^{2\gamma}\,v^{1-2\gamma}\bar{\Theta}}\right]. \tag{55}$$

The Pareto exponent α is negatively related to ε, which measures the elasticity of production to capital, via an increase of ρ^*; Remarks 6, 7 and 8 show that the relationships between α and s, ν and γ are, however, more complex.

Remark 6 *The Pareto exponent α with CES technology increases (decreases) with s if $\gamma < 0\,(\gamma > 0)$.*

Proof. The proof directly follows from the derivative of α expressed in Eq. (55) with respect to s, from which $\partial\alpha/\partial s > 0 \Leftrightarrow \gamma < 0$, given that $1-(1-\tau_k)\,\varepsilon\,(as/v)^\gamma > 0$ (see Condition (41)). ■

When $\gamma < 0$ the elasticity of substitution in production is lower than 1: the indirect effect of s on ρ^* dominates the direct effect of s on α. If technology is Cobb-Douglas, i.e. $\gamma = 0$, then $\partial\alpha/\partial s = 0$: the direct and the induced effects exactly balance. We stress that in models where factors prices are fixed α decreases with s since only the direct effect is present (see Atkinson and Harrison (1978)).

Remark 7 *The Pareto exponent α with CES technology decreases (increases) with ν if $\gamma \leq 0$ ($\gamma \in [1/2, 1]$), while if $\gamma \in (0, 1/2)$ α increases (decreases) with ν for $\nu < \tilde{v}$ ($\nu > \tilde{\nu}$), where $\tilde{\nu} = as\left[(1-\tau_k)(1-\gamma)\epsilon/(1-2\gamma)\right]^{1/\gamma}$.*

Proof. The proof directly follows from the derivative of α expressed in Eq. (55) with respect to ν. The sign of derivative is given by the sign of $(1-\gamma)(1-\tau_k)(as)^\gamma\epsilon\nu^{-\gamma} - 1 + 2\gamma$, which is always negative when $\gamma \leq 0$ (to respect Condition (41)) and always positive when $\gamma \in [1/2, 1]$. When $\gamma \in (0, 1/2)$ the derivative changes its sign in $\tilde{\nu} = as\left[(1-\tau_k)(1-\gamma)\epsilon/(1-2\gamma)\right]^{1/\gamma}$. ■

Countries with a low and declining growth rate of population could show a decline in α as well as countries with high and increasing growth rate of population. The latter result could explain the empirical evidence in Laitner (2001).

Remark 8 *The Pareto exponent α with CES technology increases (decreases) with γ if $\nu > as$ ($\nu < as$).*

Proof. Since $\partial\alpha/\partial\gamma = (\partial\alpha/\partial\rho^*)(\partial\rho^*/\partial\gamma)$ and $\partial\alpha/\partial\rho^* < 0$, it is sufficient to know the sign of $\partial\rho^*/\partial\gamma$ in order to know the sign of $\partial\alpha/\partial\gamma$. From Eq. (53) we have that $\partial\rho^*/\partial\gamma < 0 \Leftrightarrow \nu > sa$. ■

Parameter γ is directly related to the the elasticity of substitution in production; an increases in γ leads to a decrease in the Pareto exponent α if the growth rate of the population ν is sufficiently high with respect to the saving rate s. A technological change which increases the substitution between factors by increasing the return on capital can therefore have a relevant impact on the distribution of wealth.

3.2.3 Staggered Wages

We have just shown that the labour market (and taxation of labour income) does not affect the top tail of the distribution of wealth; however the working of labour market crucially affects the shape of the bottom tail of the distribution of wealth. So far we have assumed that wages perfectly respond to firms' productivity shocks, but in the real labour markets wages are generally fixed in the short run and productivity shocks are absorbed by the returns on capital (see Garcia-Peñalosa and Turnovsky (2005) for a similar point). In order to investigate the implications on the distribution of wealth of this issue we assume that all wages in the economy are set to the expected level of productivity, that is Eq. (8) is replaced by:

$$dw_j = \frac{\partial q}{\partial l_j} E\left[dA_j\right] = dw \; \forall j. \tag{56}$$

The cross-section variance of labour incomes is therefore zero: this limiting case could happen in an economy in which Trade Unions have a very strong market power, such that the bargaining on labour market is completely centralized. Wages, however, follow the marginal productivity of labour and therefore there is no unemployment. Firm j's profits are given by:

$$d\pi_j = q[k_j, l_j]dA_j - dr_j k_j - dw l_j$$

and, since in equilibrium $d\pi_j = 0$,[17] we have that:

$$dr_j = q[k_j, l_j]dA_j/k_j - dw l_j/k_j. \tag{57}$$

[17] In the model realized profits are zero because the returns on capital are residual with respect to the wages, therefore the owners of capital takes all net product not distributed to the workers.

All the results in Proposition 1 are unchanged and therefore p_i follows again Eq. (29), with the noise term η_i which satisfies $E[\eta_i] = 0$ and Eq. (30), but

$$\lim_{N\to\infty} H_{i,i'}[\vec{p}] = \left[\Delta s^2(1-\tau_k)^2\rho^2\Theta_{i,i'}\right] p_i p_{i'}. \tag{58}$$

Eq. (58) reflects the fact that now labour market is not a source of shocks for the dynamics of dynasty i's wealth. Formally, Eq. (58) obtains in the (implausible) case $\phi_{i,j} = 1/F$, where dynasties fully diversify their labour allocation across firms in the limit of an infinite economy.

Under Assumption (38) $\bar{p}$ has a deterministic dynamics given by Eq. (39); therefore also in the economy with staggered wages the results in Proposition 3 hold. In equilibrium the per capita wealth, wages and interest rate are not consequently affected by the assumption of staggered wages; however the equilibrium distribution of wealth changes. We restrict attention to the non-trivial case of incomplete markets $\bar{\Theta} > 0$.

Proposition 9 *Assume that the economy converges towards a positive and constant per capita wealth $\bar{p}^*$ and that $s[\omega^* + \tau_k\rho^*\bar{p}] > \chi$. Let $f^{SW}(p_i)$ be the equilibrium distribution of p_i with staggered wages when $N, F \to \infty$. Then:*

$$f^{SW}(p_i) = \frac{\mathcal{N}^{SW}}{a_2 p_i^{2(1+z_1/a_2)}} e^{-\left(\frac{2z_0}{a_2 p_i}\right)}, \tag{59}$$

where $\mathcal{N}^{SW}$ is a constant defined by the condition $\int_{-\infty}^{\infty} f^{SW}(p_i)\,dp_i = 1$ and z_0, z_1 and a_2 are the same as in Proposition 4.

Proof. See Appendix E. ■

At variance with the economy with perfectly flexible wages, where the distribution of wealth has support on the whole real axis, with staggered wages the distribution $f^{SW}(p_i)$ is defined only for positive wealth (i.e. $p_i > 0$). The reason is that stochastic shocks affects only returns on capital and they vanish when $p_i \to 0$ (see Eq. (58)).

For large p_i the equilibrium distribution $f^{SW}(p_i)$ follows a Pareto distribution whose exponent is equal to $\alpha^{SW} = 1 + 2z_1/a_2$, which is the same of the economy with perfectly flexible wages (see Eq. (51)). The distribution of wealth is instead markedly different for small values of p_i.[18] The intuition is that cross-section distribution of wages is crucial for the poorest dynasties: in particular, a lower volatility of wages decreases the size of the bottom tail of the distribution of wealth because the poorest dynasties

[18] More precisely, $f^{SW}(p)$ significantly deviates from the Pareto behaviour for $p \sim a_2/(2a_0)$, i.e. when the exponential factor in Eq. (59) becomes sizeable.

have an income largely dependent on wages (in the limit the cross-section volatility of wages is zero when wages are staggered). Figure 3 shows a numerical example of two distributions of wealth with CES technology.

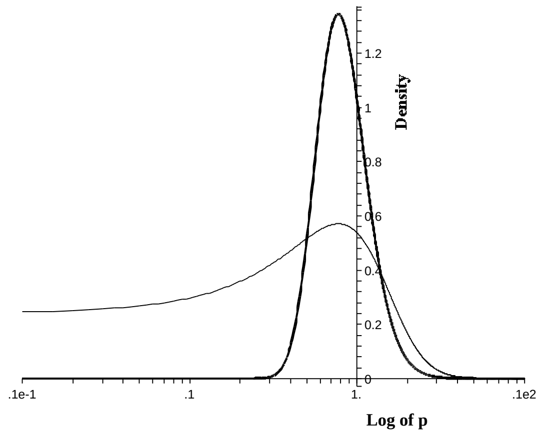

Figure 3: comparison between the distributions of wealth with perfectly flexible (thin line) and staggered (thick line) wages. The parameters assume the following values: $\nu = 0.01$,$s = 0.20$, $\chi = 0$, $\varepsilon = 0.35$, $a = 0.05$, $\gamma = -0.55$, $\tau_k = 0.20$, $\tau_l = 0.3$, $\Delta = 300$, $\Theta = 0.9$, $\Phi = 1$, $\Omega = 0$.

In Figure 3 the thin line represents the density of the distribution of wealth $f\left(p_i\right)$ with perfectly flexible wages, while the thick line represents the density $f^{SW}\left(p_i\right)$ with staggered wages. Figure 3 confirms that, when wages are staggered, the bottom tail of the distribution of wealth has a lower size and there are no dynasties with negative wealth.

3.3 Endogenous Growth Economy

Figures 4 and 5 show two economies where in equilibrium the growth of per capita wealth is caused by the ongoing accumulation of wealth (and not by an exogenous technological progress).

The worth of the analysis of the endogenous growth economy derives from the major focus on the return on capital, which determines the shape *also* of the bottom tail of the distribution of wealth; the analysis shows that the determinants of Pareto exponent are substantially the same of the exogenous growth economy, but highlights a possible relationship between the wealth inequality and the endogenous growth rate of economy.

Proposition 10 states the necessary and sufficient conditions under which in equilibrium per capita wealth grows at a positive growth rate, i.e. there is endogenous

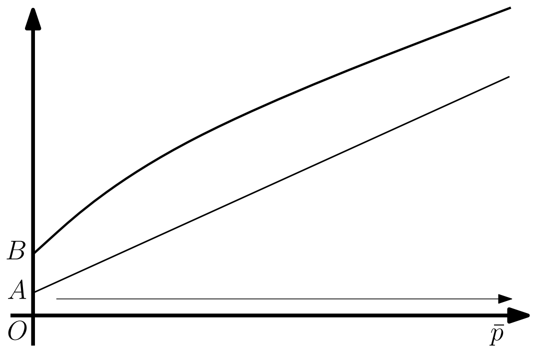

Figure 4: growing economy.

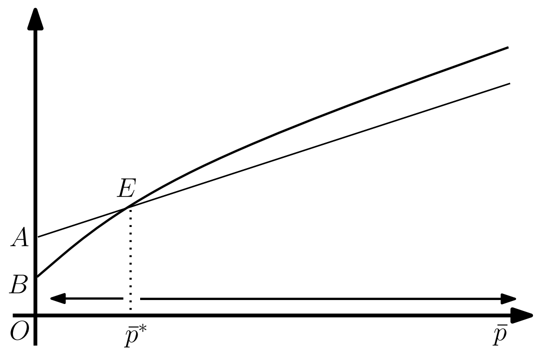

Figure 5: growing economy with a unstable equilibrium.

growth.

Proposition 10 *Assume $g(\cdot)$ satisfies Assumption (45), the dynamics of per capita wealth obeys Eq. (40) and:*

$$\lim_{\bar{p}\to\infty} g'(\bar{p}) > \frac{v}{sa}. \tag{60}$$

Then if:

$$g(0) > \frac{\chi}{sa} \tag{61}$$

in equilibrium per capita wealth will be growing at the following rate:

$$\psi^{EG} = \lim_{\bar{p}\to\infty} sag'(\bar{p}) - \nu, \tag{62}$$

indipendent of initial per capita wealth. Differently, if:

$$g(0) < \frac{\chi}{sa}, \tag{63}$$

then in equilibrium per capita wealth will be growing at constant rate ψ^{EG} if and only if the initial per capita wealth is sufficient high. When per capita wealth grows at rate ψ^{EG} the returns on factors are given by:

$$\rho^* = \lim_{\bar{p}\to\infty} ag'(\bar{p}) \text{ and} \tag{64}$$

$$\omega^* = 0. \tag{65}$$

Proof. See Appendix F. ■

In equilibrium the economy shows a behaviour similar to an AK growth model (see Barro and Sala-i-Martin (1999)), i.e. a model where marginal and average product of capital are equal and constant (or, more precisely, are bounded below). Moreover, the equilibrium interest rate ρ^* is determined only by technology and the equilibrium wage rate ω^* is zero. In equilibrium the growth rate ψ^{EG} can be written as a function of the equilibrium interest rate, i.e.:

$$\psi^{EG} = s\rho^* - \nu; \tag{66}$$

Eq. (66) shows that ψ^{EG} is independent of the flat tax rate on capital τ_k and of the diversification of dynasty i's portfolio $\bar{\Theta}$;[19] however, ψ^{EG} increases with saving rate s and with return on capital ρ^* and decreases with ν; changes in technology which increase the return on capital, therefore, cause also an increase in ψ^{EG}.

With CES technology, when $\gamma \in (0,1)$ Condition (60) is satisfied for $\varepsilon^{1/\gamma} > \upsilon/(sa)$, while when $\gamma \leq 0$ Condition (60) is never satisfied. When $\gamma \in (0,1)$ Condition (61) is satisfied for $(1-\varepsilon)^{1/\gamma} > \chi/(sa)$. With Cobb-Douglas technology endogenous growth is not possible because Condition (60) is never satisfied. Therefore, assuming $\gamma \in (0,1)$ and $\varepsilon^{1/\gamma} > \upsilon/(sa)$, we have that:

$$\psi^{EG} = sa\varepsilon^{1/\gamma} - \nu, \tag{67}$$

from which it follows that ψ^{EG} increases with ε and γ.

Proposition 11 discusses the dynamics of the dynasty i's wealth with respect to the average wealth of economy in an endogenous growth economy.

Proposition 11 *Assume $\bar{p}$ is growing at the positive rate given by Eq. (66). Let u_i be the relative per capita wealth of dynasty i, i.e. $u_i = p_i/\bar{p}$. In the long run dynasty i's relative wealth obeys the following stochastic differential equation:*

$$\lim_{t\to\infty} \frac{du_i}{dt} = s\rho^*\tau_k(1-u_i) + \tilde{\eta}_i, \tag{68}$$

where $\tilde{\eta}_i = \eta_i/\bar{p}$ is a white noise term with $E\left[\tilde{\eta}_i(t)\right] = 0$ and covariance:

$$E\left[\tilde{\eta}_i(t)\,\tilde{\eta}_{i'}(t')\right] = \delta(t-t')\,H_{i,i'}\left[\vec{u}\right], \tag{69}$$

where:

$$\lim_{t\to\infty}\lim_{N\to\infty} H_{i,i'}[\vec{u}] = \left[\Delta s^2(1-\tau_k)^2\rho^{*2}\Theta_{i,i'}\right] u_i u_{i'}.$$

[19] The assumption of constant saving rate s makes ψ^{EG} independent of τ_k. On the contrary, when saving rate is optimally chosen, s generally increases with the net return on capital $(1-\tau_k)\rho^*$; hence s decreases with τ_k.

Proof. See Appendix G. ■

In the limit $\bar{p} \to \infty$ the equilibrium wage rate converges to 0 and therefore wages do not play any role in the dynamics of relative per capita wealth of dynasty i (see Eq. (68) in Proposition 11).

Proposition 12 shows the equilibrium distribution of the relative per capita wealth u_i. Again we only focus on the non-trivial case of incomplete markets $\bar{\Theta} > 0$.

Proposition 12 *Assume that per capita wealth is growing at the rate $\psi^{EG} > 0$ and $\tau_k > 0$. Let $f^{EG}(u_i)$ be the equilibrium distribution of $u_i = p_i/\bar{p}$ when $N, F \to \infty$. Then:*

$$f^{EG}(u_i) = \frac{\mathcal{N}^{EG}}{u_i^{\alpha^{EG}+1}} e^{-(\alpha^{EG}-1)/u_i}, \tag{70}$$

where $\mathcal{N}^{EG}$ is a constant defined by the condition $\int_{-\infty}^{\infty} f^{EG}(u_i)\, du_i = 1$ and

$$\alpha^{EG} = 1 + 2\frac{\tau_k}{\Delta s(1-\tau_k)^2 \rho^* \bar{\Theta}} \tag{71}$$

is the Pareto exponent.

Proof. The proof follows the same steps reported in Appendix E taking $\mu(u_i) = s\rho^* \tau_k (1-u_i)$ and $\sigma^2(u_i) = \Delta s^2 (1-\tau_k)^2 \rho^{*2} \bar{\Theta} u_i^2$. ■

Assumption (38) is again satisfied in equilibrium since the Pareto exponent α^{EG} is always greater than 1 for $\tau_k, s > 0$; otherwise if $\tau_k = 0$ then $\alpha^{EG} = 1$ and Assumption (38) is violated. Taking the limit $\tau_k \to 0$ we do not get the same behaviour of the economy where $\tau_k = 0$: it can be shown that in the latter case u_i has a non stationary lognormal distribution.[20] Hence capital taxes do not affect growth, but have a crucial redistributive function: wealth is redistributed away from wealthy to poor dynasties by an amount proportional to aggregate wealth, so preventing the possible ever-spreading wealth levels, and stabilizing the equilibrium distribution of relative wealth.

Finally, the Pareto exponent is continuous across the transition from a stationary to an endogenously growing economy, i.e.

$$\lim_{s\rho^* - \nu \to 0^-} \alpha = \lim_{s\rho^* - \nu \to 0^+} \alpha^{EG},$$

though it has a singular behaviour in the first derivative (with respect to ν or s).

Remark 13 reports the relationships between α^{EG} and the model's parameters.

[20] Indeed, Eq. (68), with $\tau_k = 0$ and $H_{i,i'} = 0$ for $i \neq i'$, describes independent log-normal processes $u_i(t)$.

Remark 13 *The Pareto exponent α^{EG} decreases with saving rate s, return on capital ρ^*, the diversification of portfolio $\bar{\Theta}$ and it increases with τ_k; α^{EG} is, on the contrary, independent of ν.*

Proof. The proof directly follows from the derivative of α^{EG} with respect to s, ρ^*, $\bar{\Theta}$, τ_k and ν. ■

Differently from α, α^{EG} decreases with saving rate s because of the independence of ρ^* from s. The Pareto exponent α^{EG} is independent of ν because, in an economy where wealth accumulation is the source of the long-run growth, the demographic factor does not affect the return on capital (which is determined only by technology) and the level of per capita wealth.

Finally, since ψ^{EG} increases with s and ρ^*, we find an inverse relationship between growth and wealth inequality.

Remark 14 *The Pareto exponent α^{EG} and the growth rate ψ^{EG} show an inverse relationship under changes in saving rate s and/or return on capital ρ^*.*

Proof. The proof is straightforward from Eq. (66) and Remark 13. ■

For example, an economy increasing its saving rate s (or its return on capital ρ^*) should move to an equilibrium where both its growth rate and its wealth inequality (in the top tail of the distribution of wealth) are larger than before.

4 Empirical Evidence

In this section we compare the theoretical findings of the previous section with empirical evidence. Households appears the best unit of observation to test our model. We consider two datasets: the Survey of Household Income and Wealth (SHIW), which provides information on saving, income and wealth for a large sample of Italian households and the Survey of Consumer Finances (SCF), which provides, among many other variables, the net wealth of a large sample of U.S. households.[21] The comparison of these two datasets is very complex; therefore we will consider them separately.[22]

[21] In general net wealth includes all marketable assets of households. SHIW and SCF are respectively available on the following websites: http://www.bancaditalia.it/ and http://www.federalreserve.gov/pubs/oss/oss2/scfindex.html. We refer to these websites for more details on the two datasets.

[22] A first attempt to provide comparable data on these two datasets is LWS project, see www.lisproject.org/lws.htm.

The analysis aims to test if our model is able to reproduce the *qualitative* changes in the distribution of wealth. Many reasons suggest to keep our analysis at a qualitative level: i) our datasets span at most 17 years, which may be a too short period for a full convergence to equilibrium of the distribution of wealth;[23] ii) our model does not take into account many social/cultural factors affecting the distribution of wealth of a country;[24] and iii) many factors can simultaneously push in the same directions (e.g. a contemporaneous decrease in capital income taxation and increase in saving rate) and their effects are strongly nonlinear, so that disentangling the single effect of each variable on the distribution of wealth may not be feasible given the small number of waves available (9 for SHIW and 6 for SCF).

4.1 Italy

The SHIW includes data on many economic variables, among which net wealth, savings and disposable income for about 8000 Italian families for the years 1987, 1989, 1991, 1993, 1995, 1998, 2000, 2002 and 2004.[25] Brandolini et al. (2004) and Jappelli and Pistaferri (2000) present a detailed analysis of the SHIW and we refer to them for more details. By the estimate of transition matrix between the different waves of the net wealth, where states are defined by the quintiles of distribution, we calculated the *asymptotic half life*, i.e. the speed of convergence of actual distribution to the equilibrium distribution.[26] It ranges from 2.04 in 1991-1993 to 3.66 in 1993-1995, and, on average, is equal to 2.55; this means that on average 10.2 (i.e. $2.55 \times 2\times$number of lags) years are necessary to have the complete effect on the distribution of wealth of an exogenous shock (e.g. a change in the fiscal policy). Therefore the period of observation appears to be sufficiently wide.[27]

[23] Shorrocks (1975), moreover, shows that, given the type of stochastic process which describes the wealth accumulation of dynasties, the estimate of the Pareto exponent can have a non monotonous behaviour as the actual distribution converges to the equilibrium.

[24] Moreover, it is not an easy task to have a plausible estimate of the variance of the random component Δ and of the diversification of dynasties'portfolios $\bar{\Theta}$.

[25] In the SHIW the codes of the net wealth, the disposable income, the labour income, the entrepreneurial income, savings and the households' weights are respectively W, Y2, YL, YM, S2 and PESOFIT.

[26] All the statistical analysis is performed by R and all codes and datasets are available on Davide Fiaschi's website (http://www.dse.ec.unipi.it/persone/docenti/fiaschi/).

[27] Jappelli and Pistaferri (2000) estimate the asymptotic half life between 1993 and 1995 equal to 3.62, but they defined the states by the quartiles of the distribution of wealth.

4.1.1 The Estimate of Saving Function

In order to test if Eq. (11) can represent the effective consumption function of Italian households we estimate the following saving function:[28]

$$S_i = -\chi + sy_i^D - c^p p_i, \tag{72}$$

where S_i is the saving, y_i^D is the disposable income and p_i is the net wealth of dynasty i in a given year. Table 1 reports the result of the estimate of Eq. (72).

Table 1: estimate of Eq. (72) ($\hat{\chi}$ is expressed in current million liras). Source: our calculations on SHIW data

Est.\Year	1987	1989	1991	1993	1995	1998	2000	2002	2004
$\hat{\chi}$	6.62e+3	7.24e+3	9.00e+3	1.22e+4	1.46e+4	1.50e+4	1.74e+4	1.50e+4	2.47e+4
$\hat{s}$	0.48	0.51	0.55	0.58	0.61	0.60	0.64	0.58	0.74
$\hat{c}^p$	0.0080	0.0047	0.0036	0.0003	0.0067	0.0034	0.0042	0.0042	0.0084

All the estimated parameters turn out to be highly significant (we do not report t-statistics for simplicity of exposition). The estimate of $\hat{\chi}$ is increasing over time; this is likely the effect of inflation and of the increase in per capita wealth in the period.[29] The estimate of marginal saving $\hat{s}$ varies from 0.48 to 0.74, but overall $\hat{s}$ is increasing in the period 1987-2004. The estimates of marginal propensity to consume with respect to wealth $\hat{c}^p$ is more volatile, but the impact of net wealth on saving appears to be negligible (see Paiella (2004) for a similar result). The increase in $\hat{s}$ recorded after 1993 was likely caused by the severe crisis of Italian economy in 1992-1993, followed to the devaluation of lira and to the tight fiscal policy. The increase in the economic uncertainty pushed households to increase their savings (in particular their precautionary savings). In 2004 there was another large increase in the marginal saving rate, but we cannot know if such increase will be permanent or temporary.

[28] We tested the possibility of a nonlinear relation between saving, disposable income and net wealth by nonparametric methods (see Bowman and Azzalini (1997), Cap. 8). While we can reject this hypothesis for disposable income, net wealth appears to have a significant nonlinear relationship with saving (changing over time). We will ignore the latter fact because, as it is clear from the estimates reported in Table 1, the effect of net wealth on saving is negligible.

[29] The minimum consumption is likely related to a minimum standard of living, i.e. an increase in the per capita wealth should increase also χ. This intuition is confirmed by the fact that the ratio $\hat{\chi}/\bar{p}$ is nearly constant in the period 1987-2004.

Summing up Eq. (11) appears to be adequate to represents the consumption function of Italian households and overall the marginal saving rate appears to be increasing in the period 1987-2004.

4.1.2 Changes in the Top Tail of the Distribution of Wealth

Between 1987 and 2004 there was a remarkable decrease in the taxation of capital income in Italy. In particular, in recent years all capital gains are subject to a flat tax rate equal to 12.5% (the latter would correspond to τ_k of the theoretical model). Taxation on personal income was strongly progressive until the mid of 1990s, but it has been declining since 1995, with the highest marginal tax rate on income decreased from 51% to 45.5%. In Italy personal income includes both labour incomes and profits. In the second part of 1990s a credit tax, called *DIT* (dual income tax), was introduced for the firms reinvesting their profits, further decreasing the taxation on profits.

Another relevant phenomenon in the period was the decrease in the share of labour income on the aggregate product from 0.46 in 1990 to 0.42 in 2004 (see Ministero dell'Economia e delle Finanze (2005)); the possible correction for the self-employed workers does not change the scenario but only the magnitude of the decrease (from 0.67 in 1990 to 0.59 in 2004, see Jones (2003)). In the exogenous growth model with Cobb-Douglas technology the share of labour income on the aggregate product is equal to $1-\varepsilon$ (from Eqq. (52)-(52) setting $\gamma=0$),[30] which could suggest that ε increased in the period; as a consequence, the gross return on capital ρ^* increased too.

The change in the annual growth rate of population is negligible: it slightly increases from 0.05% in the period 1981-1990 to 0.17% in the period 1991-2000, but its absolute magnitude is very low.[31] Finally, the average growth rate of per capita GDP in 1991-2000 was the half of the one in 1981-1990 (i.e. 1.2% vs 2.4%).[32] The model of endogenous growth in Section 3.3, however, cannot explain such decline because in the period both s and ρ^* increased and ν was stable, suggesting that to our purposes exogenous growth model is more appropriate to represent the dynamics of Italian economy between 1987 and 2004.

The exogenous growth model predicts that an increase in saving rate s has an ambiguous effect on the Pareto exponent α; in particular, Remark 6 states that with CES technology such effect is positive (negative) if $\gamma<0$ $(\gamma>0)$, i.e. the elasticity of sub-

[30]In general the share of labour income on the aggregate output is equal to $1-\varepsilon/\left(v/sa\right)^{\gamma}$.

[31]Source: Penn World Table 6.1 (http://pwt.econ.upenn.edu/).

[32]Source: Penn World Table 6.1.

stitution in production is less (higher) than one. The lack of a reliable estimate of γ for Italian economy, however, does not allow to formulate a prediction on the effect on α of a change of s. A change in the fiscal policy in favour of capital income, on the contrary, should unambiguously decrease the Pareto exponent α. The same effect obtains under a change in technology in favour of the return on capital ρ. Overall we therefore expect a decrease in the Pareto exponent α of the distribution of net wealth of Italian households.

Figures 6 reports in the y-axis the log of cumulative density of about top 400 Italian households (5% of sample), and in the x-axis the log of normalized net wealth, i.e. the net wealth of households normalized with respect to the average net wealth (this is to control for the growth of the average net wealth).

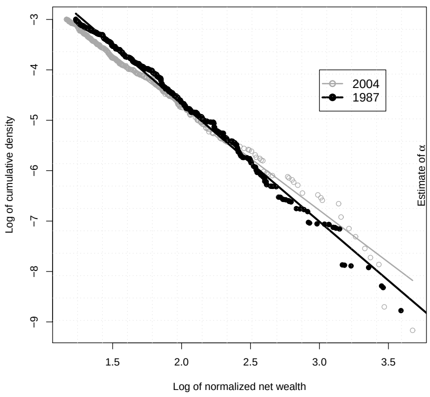

Figure 6: estimate of the distribution of net wealth of the top 5% of Italian households in 1987 and 2004. Source: our calculations on SHIW data.

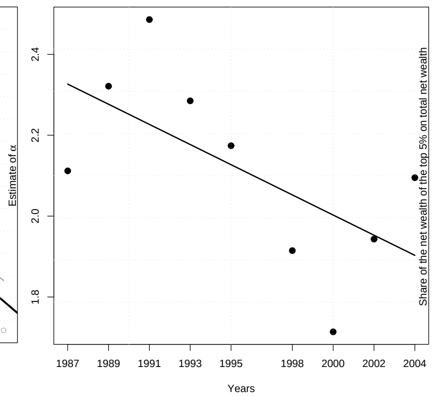

Figure 7: estimate of α by the Hill's Estimator of the top 5% of Italian households. Source: our calculations on SHIW data.

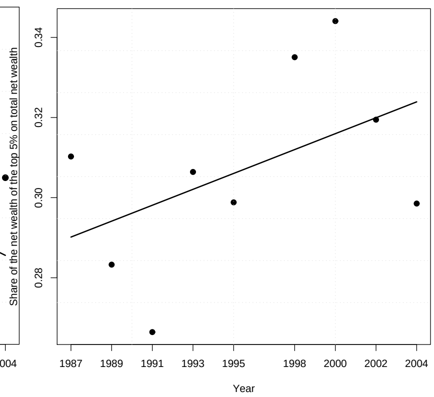

Figure 8: share of the net wealth of the top 5% of Italian households. Source: our calculations on SHIW data.

Figure 6 highlights how the relationship between the log of cumulative density and the log of normalized net wealth is approximatively linear in the top tail, which agrees with the theoretical distributions reported in Propositions 4, 9 and 10.[33] The estimate of α, $\hat{\alpha}$, corresponds to the slope of the lines reported in Figure 6. Figure 7 reports the estimates of Pareto exponent $\hat{\alpha}$ of the top 5% of Italian households for all the eight years by using the Hill's Estimator.[34]

[33] Adjusted R^2 is equal to 0.99 for both years.

[34] See Embrechts et al. (1997) for more details. We use a more general formula of the estimator than the one reported in Embrechts et al. (1997), which allows for weighted observations. In particular, given the ranked vector of households' wealth $(p_1, p_2, ..., p_N)$, where $p_1 \geq p_2 \geq ... \geq p_N$, and the vector of

Figure 7 supports the predictions of the theoretical model of a negative trend of α in the period 1987-2004; the latter is highlighted by the negative slope of the linear regression reported in Figure 7.[35]

In order to highlight the consequence of this decline in the Pareto exponent in terms of wealth inequality Figure 8 reports the share of the net wealth of top 5% on the total net wealth. The comparison of Figures 7 and 8 shows the strong correlation between $\hat{\alpha}$ and the share of the net wealth of top 5%: the linear regressions show that a decrease in α from 2.33 in 1987 to 1.90 in 2004 (the fitted values of regression) implies an increase in the share of top 5% from 0.29 in 1987 to 0.32 in 2004.

4.1.3 Change in the Bottom Tail of the Distribution of Wealth

The comparison between Propositions 4 and 9 suggests that labour market should crucially affect the bottom tail of the distribution of wealth, where workers should be the majority. Observations confirm the latter intuition: Italian households with a positive labour income have an average net wealth slightly below the average wealth $\bar{p}$ ($0.96\bar{p}$ in 1987 and $0.92\bar{p}$ in 2004), while Italian households with positive entrepreneurial income have an average net wealth slightly below $2\bar{p}$ ($1.98\bar{p}$ in 1987 and $1.94\bar{p}$ in 2004).

The Italian labour market appears to be progressively increasing its flexibility: the share of permanent jobs has decreased in favour of the share of non-permanent jobs and the market power of Trade Unions is steadily declining. The share of non-permanent jobs on the total employees, excluding the self-employed (the share of the latter on the

households' weights $(\lambda_1, \lambda_2, ..., \lambda_N)$, where $\Sigma_{j=1}^{N}\lambda_j = N$, we have that:

$$\hat{\alpha}_z = \frac{\Sigma_{j=1}^{z}\lambda_j\left(\log p_j - \log p_z\right)}{\Sigma_{j=1}^{z}\lambda_j}$$

represents the Hill's Estimator of Pareto exponent α of the distribution of the z wealthiest households (in our case z is about 400 in every year). Embrechts et al. (1997), p. 336, show that the estimate is consistent and $\sqrt{\Sigma_{j=1}^{z}\lambda_j}\left(\hat{\alpha}_z - \alpha\right) \longrightarrow N\left(0, \alpha^2\right)$.

[35]The estimated linear time-trend model is: $\hat{\alpha}\left(t\right) = 51.81 - 0.0249t$ (both coefficients are statistically significant at 10% level). The small size of the sample makes the estimate of α highly volatile from one year to the other and with high standard errors: a direct comparison between single estimates of α is therefore not very meaningful. For example, tests of equality of $\hat{\alpha}$ in 1987 and $\hat{\alpha}$ in 1998, 2000 and 2002 are respectively rejected at 10%, 1% and 10% significance level, while test of equality of $\hat{\alpha}$ in 1987 and $\hat{\alpha}$ in 2004 cannot be rejected at 20% significance level. On the contrary tests of equality of $\hat{\alpha}$ in 1989 and $\hat{\alpha}$ in 1998, 2000, 2002 and 2004 are respectively rejected at 1%, 1%, 1% and 10% significance level. Hypothesis testing follows the bootstrap procedure described in Efrom and Tibshirani (1993), p. 224 with 1000 bootstraps. Hill's plots show that the estimate of α is convergent for almost all years.

total employees is stable around 0.27-0.28 over the period 1993-2003) increased from 6.2% in 1993 to 9.7% in 2003.[36]

In the same period the net union membership strongly decreased: the share of memberships, excluding self employed and retired, on the total labour force declined from 36.5% in 1985 to 30.9% in 1997 (see Golden et al. (2004)). Both phenomena should lead to an increase in the cross-section variance of labour incomes. Figure 9 corroborates this intuition.

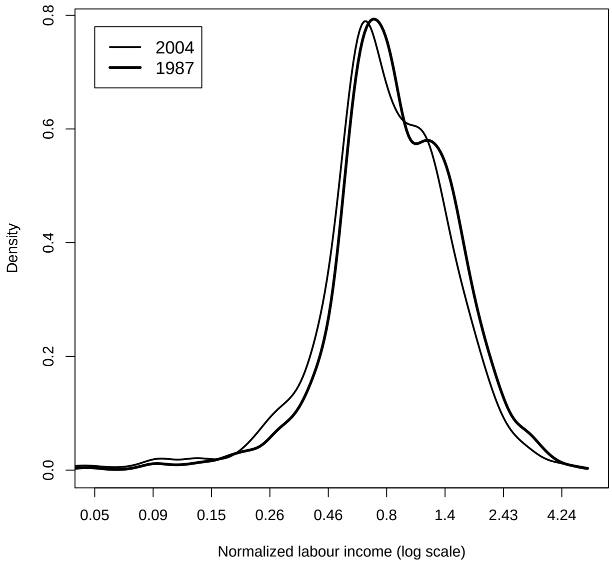

Figure 9: estimate of the distribution of labour incomes in 1987 (thick line) and 2004 (thin line) in Italy. Source: our calculation on SHIW data.

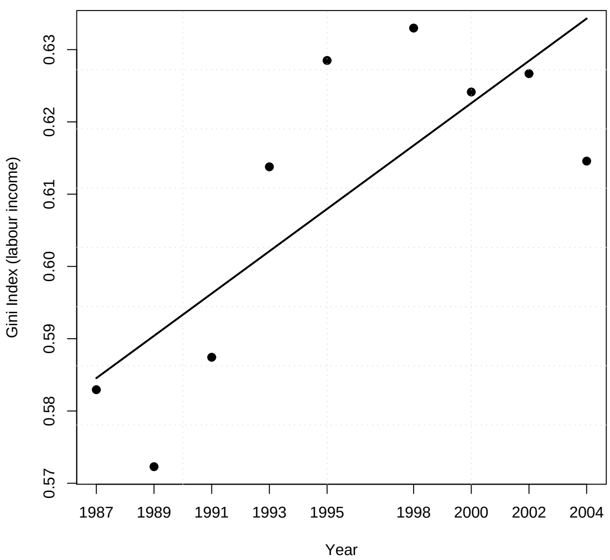

Figure 10: Gini index of the distribution of labour income in Italy. Source: our calculations on SHIW data.

Figure 9 reports the estimated distribution of the log of normalized (gross) labour incomes of Italian households in 1987 and 2004 (in both periods observations are normalized to the average).[37] We observe an increase in the size of the bottom tail and, in general, an increase in the variance of distribution. The latter is confirmed by the increase in the Gini index of the distribution of labour incomes from 0.58 in 1987 to 0.62 in 2004 (see Figure 10).[38]

[36]Source: Ministero dell'Economia e delle Finanze (2005).

[37]For all the kernel density estimations we used the package "sm" with the standard setting, i.e. Gaussian kernel and the normal optimal smoothing bandwidth; see Bowman and Azzalini (1997) for more details.

[38]The estimated linear time-trend model is: *Gini Index* $(t) = -5.23 + 0.003t$ (both coefficients are statistically significant at 5% level). Test of equality of the estimated Gini indeces in 1987 and in 2004 is rejected at 5% significance level (hypothesis testing follows again the bootstrap procedure described

The comparison between Propositions 4 and 9 suggests that an increase in the cross-section variance of the distribution of labour incomes should imply an increase in the size of the bottom tail of the wealth distribution (see Figure 3). Figures 11 and 12 respectively report the kernel density estimations of the distribution of net wealth in 1987 and 2004 (only for Italian households with positive net wealth) and the share of Italian households with negative net wealth for all available years.

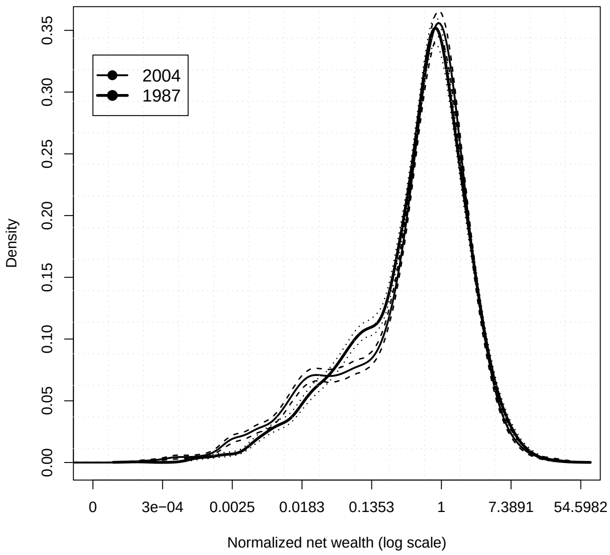

Figure 11: comparison between the distribution of net wealth in 1987 (thick line) and in 2004 (thin line) (only Italian households with positive net wealth). The dotted and dashed lines are, respectively, the 95% confidence intervals of the estimates in 1987 and in 2004. Source: Source: our calculations on SHIW data.

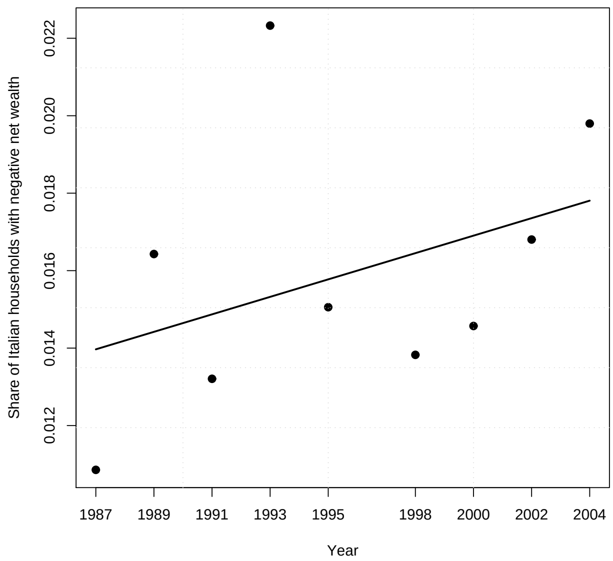

Figure 12: share of Italian households with negative net wealth. Source: our calculations on SHIW data.

Figure 11 shows that the estimates of the densities in 1987 and 2004 are statistically different in the low tail.[39] Figure 12 shows that the share of Italian households with negative net wealth is slightly increasing in the period.[40] Overall the evidence support

in Efrom and Tibshirani (1993) with 1000 bootstraps). Finally, Brandolini et. al. (2001) find a large increase in the earnings dispersion of the Italian households in the early 1990s.

[39]The 95% confidence intervals reported in Figure 11 are calculated by a bootstrap procedure suggested in Bowman and Azzalini (1997), p. 44, with 500 bootstraps.

[40]The estimated linear time-trend model is: *Share of households with negative net wealth* $(t) = -0.44 + 0.0002t$ (both coefficients are, however, not statistically significant at 10% level).

the prediction that the size of the bottom tail of the distribution of wealth increased in the period 1987-2004 as a consequence of the increase in the flexibility of Italian labour market.

To conclude, we observe that the changes in the top and bottom tails of the distribution of wealth could suggest an increase in the overall inequality of the distribution of wealth; however, inspection of Figure 11 shows a strong increase in the density around the mean (i.e. 1 in Figure 11), which means less inequality in the middle of the distribution. In a synthetic index of inequality the latter effect may outweigh the increase in the inequality on the two tails of distribution. Indeed the Gini index of the distribution of wealth decreased from 0.62 in 1987 to 0.60 in 2004.[41]

4.2 U.S.

We take the observations on the net wealth of U.S. households from the Survey of Consumer Finances (SCF); the following years are available: 1989, 1992, 1995, 1998, 2001 and 2004. The number of U.S. households included in the SCF has increased from 3143 in 1989 to 4442 in 2004. Observations on the net wealth and income are easily available on the SCF's website, but unfortunately neither savings nor earnings are available.[42] Also panel information on the households in the sample are not included. We refer to Wolff (2004) for more details on the SCF.[43]

We partially try to fill this gap by using data on labour incomes reported in PSID.[44] PSID also reports data on net wealth. However, the wealthy households are strongly underrepresented, so that we prefer the SCF for studying the distribution of wealth.

[41]In particular, Gini index of the distribution of wealth is equal to 0.62 in 1987, 0.59 in 1989, 0.59 in 1991, 0.63 in 1993, 0.61 in 1995, 0.63 in 1998, 0.63 in 2000, 0.62 in 2002 and 0.60 in 2004. Test of equality of the estimated Gini indeces in 1987 and in 2004 is rejected at 1% significance level.

[42]The variable "SAVING" in the SCF cannot be used in the estimate of the saving function because its definition does not match the standard definition of saving, it is not calculated as disposable income minus consumption expenditure.

[43]In our calculations we use the sample weights reported in SCF's website and this determines the differences with Wolff (2004)'s results; the main difference is for the weights of low-wealth households, which he says to be underrepresented in the sample SCF. These differences are relevant both in magnitude, e.g. Gini index of the distribution of wealth is equal to 0.81 in 2001 in our calculations while it is equal to 0.83 in Wolff (2004) and over time, e.g. Gini index is constant between 1989 and 2001 in Wolff (2004) (0.83 in 1989 vs 0.83 in 2001) but increasing in our calculations (respectively 0.79 vs 0.81). In this regard Davies and Shorrocks (1999), in suggesting to use SCF for the analysis of U.S. distribution of wealth, warn about the controversy on which weights must be used in the estimate.

[44]We take data directly from website http://psidonline.isr.umich.edu/.

The panel framework of PSID, however, allows to estimate the speed of convergence of the actual distribution of wealth to the equilibrium.[45] The estimated asymptotic half life is equal to 3.46 in 1999-2001 and 3.81 in 2001-2003, i.e. on average it needs 14.5 years to have the complete impact on the distribution of wealth of a exogenous shock. The period for which observations are available, therefore, appears to be sufficiently wide.

4.2.1 Change in the Top Tail of the Distribution of Wealth

Table 2 highlights that the average tax rate on income in U.S. has slightly decreased over the period 1985-2003, but the main benefits are for the top 1% income people (about -6.5%), while top 25% and the decrease for the top 50% income people has been milder (about -2.5%); morevoer, the major changes happened at the end of 1980s.

Table 2: average tax rates on income in 1985 and in 2003 in U.S.. Source: IRS (http://www.irs.gov/taxstats/)

Year	Total	Top 1%	Top 5%	Top 10%	Top 25%	Top 50%
1985	13.89	30.87	24.07	21.34	17.80	15.59
2003	11.90	24.31	20.74	18.49	15.38	13.35

Also the tax on capital income decreased in the period. The average and the marginal federal tax rates on the corporate profits respectively decreased from 27.43% in 1994 to 25.58% and from 26.82% in 1994 to 23.63 in 2002.[46] The decline in the taxation of corporate profits is particularly strong for the marginal tax rate. The marginal tax rate is, moreover, lower than the average tax rate; this is the effect of higher tax credits to the biggest corporations. Finally, the Jobs and Growth Tax Relief Reconciliation Act of 2003 provides further concessions for U.S. households and a major cut in the tax rates on capital gains and dividends.

The average saving rates of U.S. households has fallen in the period 1985-2003, starting from an already low initial level, from 9.0% in 1985 to 1.8% in 2004.[47]

[45]In the estimate we use the following variables (we report the PSID code): ER417, S517 and S617 (U.S. households' net wealth in 1999, 2001 and 2003 respectively) and FCWT99, ER20394 and ER24179 (sample weights for 1999, 2001 and 2003 respectively)

[46]Source: our calculations on IRS data. Average tax rate is the ratio between the total income after tax credit and income subject to tax, while the marginal tax rate is the same ratio calculated for the corporations in the highest class in terms of assets (over 2.500 millions of dollars).

[47]Source: BEA (http://www.bea.gov/).

The share of labour income on the aggregate product did not change appreciably in the period 1985-2004, being stable around 0.67 (see Jones (2003)). There is virtually no change in the annual growth rate of population: 0.93% in the period 1981-1990 and 0.97% in the period 1991-2000.[48] Finally, the average growth rate of per capita GDP in 1991-2000 is about the same as the one in 1981-1990 (i.e. 2.2% vs 2.3%).[49]

Our theoretical model suggests that the decrease in the tax rate on capital income should lead to a decrease in the Pareto exponent α. We have no data on the marginal saving rate s of U.S. households, but we conjecture that the latter is decreasing in the period on the basis of the dynamics of the average saving rate. This decline in s should induce a further decrease in α, since empirical analysis usually found an elasticity of substitution in production lower than 1 for U.S. (see, e.g. Chirinko (1993)). Overall, the model therefore predicts a decrease in α.

In the estimate of α we consider top 5% of U.S. households, excluding top 0.5%. The extreme top tail, indeed, appears to be underrepresented and this could bias upward the estimate of α.[50] For example, in order to respect the privacy of the richest U.S. households, SCF does not explicitly consider the 400 wealthiest people included in the Forbes list; the total wealth of the latter account for 1.5% in 1989 and 2.2% in 2001 of total U.S. wealth (see Kennickell (2003), p. 3). For the wealthiest people we refer to Klass et al. (2006), who show how the distribution of wealth for the people in the Forbes list follows a Pareto distribution, whose Pareto exponent decreased from 1.6 in 1988 to 1.2 in 2003 (see also Castaldi and Milakovic (2006)).

Figure 13 shows that the top tail of the distribution of net wealth follows a Pareto law in 1989 and in 2004. Figure 14 highlights that the decrease in the estimate of Pareto exponent $\hat{\alpha}$, which decreases from 1.38 in 1989 to 1.13 in 2004; this agrees with Klass et al. (2006)'s results.[51] Figure 15 confirms the inverse relationship between $\hat{\alpha}$ and the size of top tail of the distribution: a decline in $\hat{\alpha}$ from 1.38 in 1989 to 1.13 in 2004 corresponds an increase in the share of top 5% from 0.54 to 0.57.

[48]Source: Penn World Table 6.1.

[49]Source: Penn World Table 6.1.

[50]Comparable results are obtained by excluding 0.1% of top tail. We choose to exclude 0.5% of top tail because regressions presents an higher adjusted R^2.

[51]The estimated linear time-trend model is: $\hat{\alpha}(t) = 35.87 - 0.017t$ (both coefficients are statistically significant at 1% level). Tests of equality of $\hat{\alpha}$ in 1987 and in 2001 and 2004 are respectively rejected at 10% and 15% significance level, while tests of equality of $\hat{\alpha}$ in 1987 and in 1992, 1995 and 1998 cannot be rejected at 20% significance level. Hill's plots show that the estimate of α is convergent for almost all years.

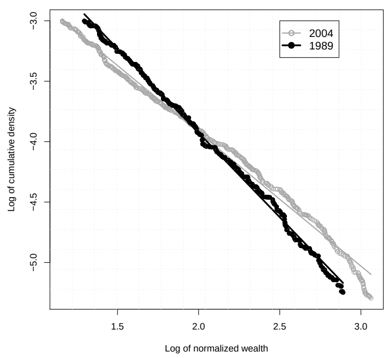

Figure 13: estimate of the distribution of net wealth of the top 5% of U.S. households in 1989 and in 2004. Source: our calculations on SCF data.

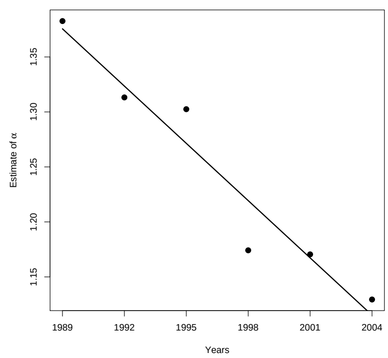

Figure 14: estimate of α by the Hill's Estimator of the top 5% of U.S. households. Source: our calculations on SCF data.

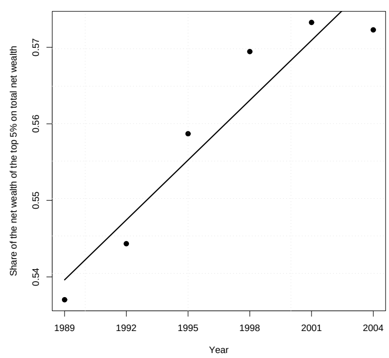

Figure 15: share of the net wealth of the top 5% of U.S. households. Source: our calculations on SCF data.

4.2.2 Change in the Bottom Tail of the Distribution of Wealth

As we discussed above, according to our model the working of labour market crucially affects the shape of the bottom tail of the distribution of wealth. In the U.S. labour market the share of non-permanent jobs has not changed significantly in the last 15 years, but the net union density sharply declined from 17.2 in 1985 to 13.5 in 2000 (see Golden et al. (2004)). Figure 9 reports the estimate of distribution of the log of normalized (gross) labour incomes in 1989 and 2002 and Figure 17 the Gini index of the distribution of labour incomes in 1981, 1989, 1996 and 2002.[52]

The peak of the estimated density shows a clear shift below 1 and a consequent increase in the size of the bottom tail of distribution. The increase in the cross-section variance of the distribution is confirmed by Figure 17; the latter shows the increase from 0.43 in 1989 to 0.46 in 2002 of the Gini index of the distribution of labour incomes.[53] Therefore the model predicts an increase in the size of the bottom tail of the distribution of wealth.

Figure 18 reports the estimated distribution of net wealth in 1989 and 2001 (we

[52]In the estimate we consider the labour incomes of the head of household (in PSID database ER24116, ER12080, V18878, V17534 and V8690 are respectively the codes for labour incomes in 2002, 1996, 1989, 1988 and 1981 and ER24179, ER12084, V18945, V17612 and V8727 for sample weights in 2002, 1996, 1989, 1988 and 1981).

[53]The estimated linear time-trend model is: $\textit{Gini Index}\,(t) = -4.34 + 0.0024t$ (both coefficients are statistically significant at 5% level). Test of equality of the estimated Gini indeces in 1989 and in 2002 is rejected at 5% significance level.

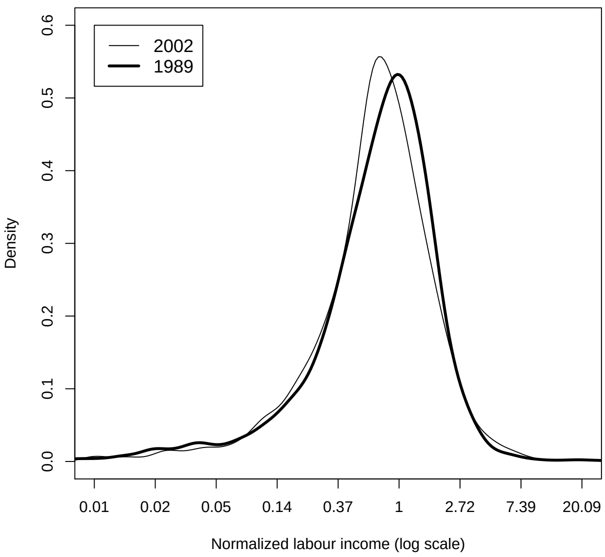

Figure 16: estimate of the distribution of labour incomes in U.S. in 1989 and in 2002. Source: our calculations on PSID data.

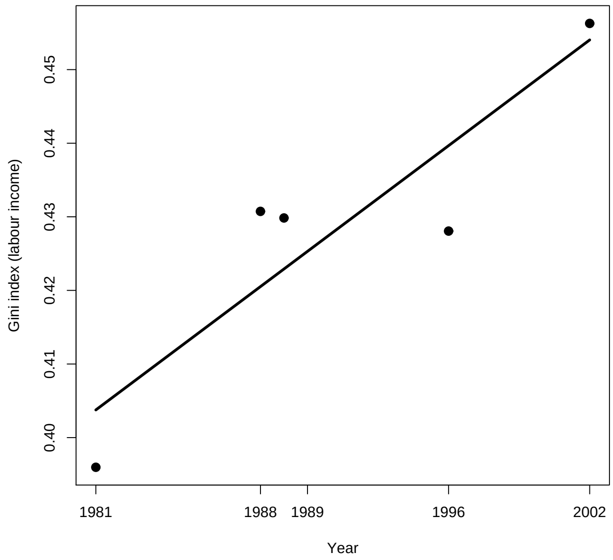

Figure 17: Gini index of the distribution of labour incomes in U.S. in 1981-2002. Source: our calculations on PSID data.

include only U.S. households with positive net wealth), while Figure and 19 the share of U.S. households with negative net wealth.

Figure 18 shows that the estimates of the densities in 1989 and 2004 are statistically different in the lower tail;[54] in particular, the share of U.S. households with a positive net wealth below the average appears to be increased (the two distributions cross in correspondence of about 0.38 in Figure 19). Figure 19 shows that the share of U.S. households with negative net wealth, on the contrary, is almost constant between 1989 and 2004.[55]

Overall, the changes in the top and bottom tails of the distribution of net wealth in U.S. suggest that the wealth inequality has increased. The downward shift in the peak of the distribution of net wealth also contributes to the increase in wealth inequality. Gini index of the distribution of net wealth confirms such intuition: it is increases from 0.79 in 1989 to 0.81 in 2004.[56]

[54]The 95% confidence intervals reported in Figure 11 are again calculated by the bootstrap procedure suggested by Bowman and Azzalini (1997) with 500 bootstraps.

[55]This is confirmed by the estimated linear time-trend model: *Share of households with negative net wealth* $(t) = -0.1091 + 0.000091t$ (both coefficients are not statistically significant at 10%).

[56]In particular, Gini index of the distribution of wealth is equal to 0.79 in 1989, 0.79 in 1992, 0.79 in 1995, 0.80 in 1998, 0.81 in 2001 and 0.81 in 2004. Test of equality of the estimated Gini indeces in 2004 and in 1989 is rejected at 15% significance level.

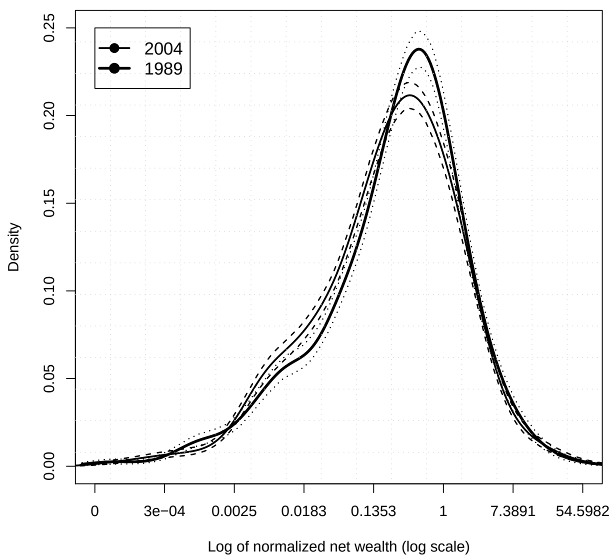

Figure 18: comparison between the distributions of net wealth in 1989 and in 2001 (only U.S. households with positive net wealth). The dotted and dashed lines are, respectively, the 95% confidence intervals of the estimates in 1989 and in 2004. Source: our calculations on SCF data

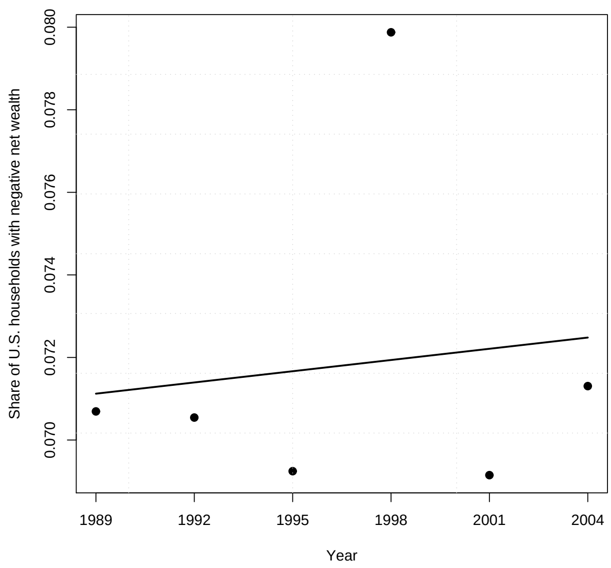

Figure 19: share of U.S. households with negative net wealth. Source: our calculations on SCF data

5 Conclusions and future research

This paper is a first step toward a theory of the distribution of wealth. We characterize the equilibrium distribution of wealth in an economy with a large number of firms and dynasties, who interact through the capital and the labour markets. Under incomplete markets, the top tail of the equilibrium distribution of wealth is well-represented by a Pareto distribution, whose exponent depends on the saving rate, on the net return on capital, on the growth rate of the population, on the tax on capital income and on the degree of diversification of portfolios. The latter is meant to reflect frictions and transaction costs in the capital market (with risk-adverse dynasties). On the other hand, the bottom tail of the distribution mostly depends on the working of the labour market: a labour market with a centralized bargaining where workers do not bear any risk determines a lower wealth inequality.

Our framework neglects important factors which have been shown to have a relevant impact on the distribution of wealth, such as, for example, the possible optimizing behaviour of dynasties with respect to choice on consumption/saving, the age structure of the population, inheritance patterns and marriage (see Davies and Shorrocks (1999)). Moreover, our results are relative to the equilibrium distribution of wealth: the analysis of out-of-equilibrium behaviour seems a necessary extension, also to take into account the speed of convergence of the actual distribution to its equilibrium and its possible nonmonotonic behaviour (see Atkinson and Harrison (1978), p. 227). Benhabib and Bisin (2007)'s model is a step in this direction. In this respect the Italian observations show deviations from equilibrium over time scale of few years. We conjecture that the latter are related to the possible soaring in the real estate prices, yet another effect not considered in our model.

The lack of space has limited our analysis in many stimulating directions. We do not deepen the relationship between the distribution of wealth and the distribution of income. Heuristically we can argue that in our model the wealth inequality is always higher than the income inequality because generally only a small share of current income derives from wealth (empirically plausible returns on wealth are well below 10%); the other part of income derives from wages and from government transfers, which are more equally distributed across dynasties. Moreover, in contrast with the empirical evidence, we assume that diversification of portfolios, as well as the return on investments of a dynasty are independent of its level of wealth. A dependence of the portfolios diversification and the return on investment on the level of wealth has been argued by Shorrock (1988) to have potential consequences on the top tail of distribution (see also Arrow (1996)).

Two further extensions look promising. The first is related to an economy where labour market and capital market have different speeds of adjustment to equilibrium. It seems realistic to assume that labour market adjusts at a slower pace than the capital market. In such a situation, productivity shocks would impact mostly the capital market. In this sense, the case of staggered wages considered here might be thought of as the extreme case where wages evolve over infinitely slower time scales. Naively speaking, we expect that a slow speed of adjustment in labour market decreases the cross-section variance of wages and hence reduces inequalities in the bottom tail of the distribution of wealth. However, full account of these issues would entail dealing with situations where firms are constrained in the choice of the factors of production, with consequent underutilization of factors (i.e. unemployment).

The second interesting extension is an economy in which aggregate wealth exhibits a stochastic behaviour. In the light of our findings, the latter behaviour can arises because of correlations in productivity shocks, which were neglected here, because dynasties concentrate their investments in few firms/assets (due to, e.g., market incompleteness) or because the number of firms/assets is much smaller than the number of dynasties. This extension would draw a theoretical link between the dynamics of the distribution of wealth, the distribution of firm size and business cycle (see Gabaix (2008)).

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A Proof of Proposition 1

Given Eqq. (12), (18) and (19) we have

$$dp_i = s\left\{\left[\sum_{j=1}^{F}(1-\tau_k)\,dr_j\theta_{i,j}p_i(t) + (1-\tau_l)\,dw_j\phi_{i,j}l_i\right] + \right.$$
$$\left. + \frac{1}{N}\sum_{i'=1}^{N}\left[\sum_{j=1}^{F}\tau_k dr_j\theta_{i',j}p_{i'}(t) + \tau_l dw_j\phi_{i',j}l_{i'}\right]\right\} - d\bar{c} - dn_ip_i. \tag{73}$$

Taking a continuum time limit the dynamics of p_i is described by the Langevin equation (see Gardiner (1997)):

$$\frac{dp_i}{dt} = F_i[\vec{p}] + \eta_i, \tag{74}$$

where $E[\eta_i(t)] = 0$ and the covariance of η_i is given by:

$$E[\eta_i(t)\,\eta_{i'}(t')] = H_{i,i'}[\vec{p}]\,\delta(t-t').$$

In Eq. (74)

$$F_i[\vec{p}] = \lim_{dt\to 0}\frac{E[dp_i]}{dt} \text{ and} \tag{75}$$
$$H_{i,i'}[\vec{p}] = \lim_{dt\to 0}\frac{1}{dt}E[(dp_i - E[dp_i])(dp_{i'} - E[dp_{i'}])]. \tag{76}$$

From Eq. (73) we have that:

$$E[dp_i] = s\left[(1-\tau_k)\,drp_i + (1-\tau_l)\,dwl_i + \tau_k dr\bar{p} + \tau_l dw\bar{l}\right] - d\bar{c} - np_i, \tag{77}$$

which together with Eq. (75) and Eqq. (21)-(24) leads to:

$$F_i[\vec{p}] = s\left[(1-\tau_k)\,\rho p_i + (1-\tau_l)\,\omega l_i + \tau_k\rho\bar{p} + \tau_l\omega\bar{l}\right] - \chi - \nu p_i.$$

In order to compute $H_{i,i'}[\vec{p}]$ note that from Eqq. (73) and (77):

$$dp_i - E[dp_i] = s\sum_{j=1}^{F}\left\{(1-\tau_k)\,\rho p_i\theta_{i,j} + (1-\tau_l)\,\omega l_i\phi_{i,j} + \right. \tag{78}$$
$$\left. + \frac{1}{N}[\tau_k\rho + \tau_l w/\lambda]\,k_j\right\}d\zeta_j$$

from which:

$$H_{i,i'}[\vec{p}] = \Delta s^2\left\{(1-\tau_k)^2\rho^2 p_ip_{i'}\Theta_{i,i'} + (1-\tau_l)^2\omega^2 l_il_{i'}\Phi_{i,i'} + \right.$$
$$+ (1-\tau_k)(1-\tau_l)\rho\omega\left[p_il_{i'}\Omega_{i,i'} + l_ip_{i'}\Omega_{i',i}\right] +$$
$$+ \frac{\tau_k\rho + \tau_l\omega/\lambda}{N}\left[(1-\tau_k)\rho(p_i\vartheta_i + p_{i'}\vartheta_{i'}) + (1-\tau_l)\omega(l_i\varphi_i + l_{i'}\varphi_{i'})\right] +$$
$$\left. + \frac{[\tau_k\rho + \tau_l\omega/\lambda]^2}{N^2}\sum_{j=1}^{F}k_j^2\right\},$$

where we used Definition (27) and the parameters $\Theta_{i,i'}$, $\Omega_{i,i'}$ and $\Phi_{i,i'}$ are defined in Eq. (20) and ϑ_i, φ_i in Eq. (31).

QED

B Proof of Proposition 2

From Eq. (78) we have that:

$$\bar{\eta} dt = \frac{1}{N} \sum_{i=1}^{N} \left[dp_i - E\left(dp_i\right) \right] =$$
$$= s \sum_{j=1}^{F} \left\{ \left(1-\tau_k\right) \rho \frac{\sum_{i=1}^{N} p_i \theta_{i,j}}{N} + \left(1-\tau_l\right) \omega \frac{\sum_{i=1}^{N} l_i \phi_{i,j}}{N} + \frac{1}{N} \left[\tau_k \rho + \tau_l \omega / \lambda \right] k_j \right\} d\zeta_j =$$
$$= \frac{1}{N} \left\{ s \left(\rho + \omega/\lambda\right) \sum_{j=1}^{F} k_j d\zeta_j \right\},$$

since $k_j = \sum_{i=1}^{N} p_i \theta_{i,j}$, $l_j = \sum_{i=1}^{N} l_i \phi_{i,j}$ and $k_j = \lambda l_j \; \forall j$. In order to have no stochastic fluctuation in $\bar{p}$ we need $E\left[\bar{\eta} dt\right] = 0$ and $E\left[\left(\bar{\eta} dt\right)^2\right] = 0$ since $\forall \epsilon > 0$

$$P\left(\left|\bar{\eta} dt\right| > \epsilon\right) \leq \frac{E\left[\left(\bar{\eta} dt\right)^2\right]}{\epsilon^2} \tag{79}$$

for the Chebyshev's inequality. It is straightforward to see that:

$$\lim_{N \to \infty} E\left[\bar{\eta} dt\right] = 0$$

and that:

$$\lim_{N \to \infty} E\left[\left(\bar{\eta} dt\right)^2\right] =$$
$$= \lim_{N \to \infty} \frac{1}{N^2} \left\{ s^2 \left(\rho + \omega/\lambda\right)^2 \Delta \sum_{j=1}^{F} k_j^2 \right\} dt = 0 \Leftrightarrow \lim_{N \to \infty} \frac{1}{N^2} \sum_{j=1}^{F} k_j^2 = 0.$$

Moreover,

$$\frac{\sum_{j=1}^{F} k_j^2}{N^2} = \frac{\sum_{i=1}^{N} p_i p_{i'} \Theta_{i,i'}}{N^2} \leq \frac{\sum_{i=1}^{N} p_i^2 \Theta_{i,i'}}{N^2} = \frac{\sum_{i=1}^{N} p_i^2 \sum_{j=1}^{F} \theta_{i,j} \sum_{i'=1}^{N} \theta_{i',j}}{N^2} \leq \bar{\theta} \bar{p}^2 \sum_{i=1}^{N} \left(\frac{p_i}{P}\right)^2, \tag{80}$$

where in the first inequality of Eq. (80) we used the Cauchy's inequality (see Hardy et al. (1954)):

$$\sum_k a_k b_k \leq \sqrt{\sum_k a_k^2 \sum_k b_k^2}$$

setting $k = i, i'$ and $a_k = p_i \sqrt{\Theta_{i,i'}}$ and $b_k = p_{i'} \sqrt{\Theta_{i,i'}}$, while in the last passage of Eq. (80) we use $\sum_{j=1}^{F} \theta_{i,j} = 1$. Assumption (37) therefore ensures that $\lim_{N\to\infty} E\left[(\bar{\eta} dt)^2\right] = 0$.

QED

C Proof of Proposition 3

From Eq. (40) for large value of $\bar{p}$ we have that Condition (41) ensures that $d\bar{p}/dt < 0$; in fact:

$$\lim_{\bar{p}\to\infty} \frac{d\bar{p}}{dt} < 0 \Leftrightarrow \lim_{\bar{p}\to\infty} \frac{g(\bar{p})}{\bar{p}} = \lim_{\bar{p}\to\infty} g'(\bar{p}) < \frac{\nu}{sa}.$$

Condition (42) states that in $\bar{p} = 0$ $d\bar{p}/dt > 0$. Since $g(\cdot)$ is continuous, always increasing and concave, then there exists only one value of $\bar{p}$, $\bar{p}^*$, such that $d\bar{p}/dt = 0$, i.e.:

$$sag(\bar{p}^*) = \chi + v\bar{p}^*.$$

On the contrary Condition (43) states that in $\bar{p} = 0$ $d\bar{p}/dt < 0$. This means that $\bar{p} = 0$ is an equilibrium. Condition (44) states that two other equilibria exist (see Figure 2). It is straightforward to shows that the low equilibrium is unstable, while the high equilibrium is locally stable. Eqq. (46) and (47) are directly derived by Eqq. (15) and (16), taking into account Eqq. (18) and (19).

QED

D Proof of Proposition 4

In the infinite economy, when per capita wealth converges to its equilibrium level, dp_i/dt depends on the wealth of all the other dynasties only through the equilibrium per capita wealth $\bar{p}^*$. Hence the determination of the marginal distribution of p_i reduces to a single dynasty problem which, under Assumptions (32)-(35), is given by:

$$\frac{dp_i}{dt} = \mu(p_i) + \eta_i; \tag{81}$$
$$E[\eta_i(t)\eta_i(t')] = \sigma^2(p_i)\,\delta(t - t')$$

where, from Proposition 1, we have:

$$\mu(p_i) = z_0 - z_1 p_i; \tag{82}$$
$$\sigma^2(p_i) = \lim_{N\to\infty} H_{i,i}[\vec{p}] = a_0 + a_1 p_i + a_2 p_i^2, \tag{83}$$

with:

$$\begin{aligned}
z_0 &= s\left(\omega^* + \tau_k \rho^* \bar{p}\right) - \chi; \\
z_1 &= \nu - s\left(1 - \tau_k\right)\rho^*; \\
a_0 &= \Delta s^2 \left(1 - \tau_l\right)^2 \omega^{*2} \bar{\Phi}; \\
a_1 &= 2\Delta s^2 (1 - \tau_k)(1 - \tau_l)\rho^* \omega^* \bar{\Omega} \text{ and} \\
a_2 &= \Delta s^2 \left(1 - \tau_k\right)^2 \rho^{*2} \bar{\Theta}.
\end{aligned}$$

The last two terms in braces in the expression of $H_{i,i}\left[\vec{p}\right]$ in Proposition 1 vanish in the limit $N \to \infty$.

The Fokker-Planck equation corresponding to Eq. (81) is given by (see Gardiner (1997), p. 118):

$$\frac{\partial \tilde{f}\left(p_i, t\right)}{\partial t} = -\frac{\partial}{\partial p_i}\left[\mu\left(p_i\right) - \frac{1}{2}\frac{\partial \sigma^2\left(p_i\right)}{\partial p_i}\right]\tilde{f}\left(p_i, t\right).$$

Since $z_0 > 0$, in equilibrium $\partial \tilde{f}\left(p_i, t\right)/\partial t = 0$, that is the equilibrium distribution of wealth $f\left(p_i\right)$ must satisfy:

$$\mu\left(p_i\right) f\left(p_i\right) = \frac{1}{2}\frac{\partial}{\partial p_i}\left[\sigma^2\left(p_i\right) f\left(p_i\right)\right].$$

Take $\varphi\left(p_i\right) = \sigma^2\left(p_i\right) f\left(p_i\right)$, then:

$$\frac{\partial \varphi\left(p_i\right)}{\partial p_i} = \frac{2\mu\left(p_i\right)\varphi\left(p_i\right)}{\sigma^2\left(p_i\right)},$$

that is:

$$\varphi\left(p_i\right) = B e^{2\int_{-\infty}^{p_i} dx_i \mu(x_i)/\sigma^2(x_i)},$$

where B is a constant; finally (see Gardiner (1997), p. 124):

$$f\left(p_i\right) = \left[\frac{B}{\sigma^2\left(p_i\right)}\right] e^{2\int_{-\infty}^{p_i} dx_i \mu(x_i)/\sigma^2(x_i)}. \tag{84}$$

With complete markets, i.e. $\bar{\Theta}, \bar{\Omega} = \bar{\Phi} = 0$, the variance of p_i is zero, i.e. $\sigma^2\left(p_i\right) = 0$ (see Eq. (83)), and therefore the equilibrium distribution collapses towards the mean, i.e. $p_i = \bar{p}\ \forall i$ in the equilibrium.

With complete financial markets, i.e. $\bar{\Theta}, \bar{\Omega} = 0$ and $\bar{\Phi} > 0$, the integral in Eq. (84) is given by:

$$\int dx_i \frac{\mu\left(x_i\right)}{\sigma^2\left(x_i\right)} = \frac{z_0}{a_0}p_i - \frac{z_1}{a_0}p_i^2,$$

from which:

$$f\left(p_i\right) = \mathcal{N} e^{-\frac{\left(z_0 - z_1 p_i\right)^2}{z_1 a_0}},$$

where $\mathcal{N}$ is such that $\int_{-\infty}^{\infty} f(p_i)\, dp_i = 1$.

With incomplete markets, i.e. $\bar{\Theta}, \bar{\Omega}, \bar{\Phi} > 0$, the integral in Eq. (84) is given by:

$$\int dx_i \frac{\mu(x_i)}{\sigma^2(x_i)} = -\left(\frac{z_1}{2a_2}\right) \ln \sigma^2(x_i) + 2\left(\frac{z_0 + \frac{z_1 a_1}{2a_2}}{\sqrt{4a_0a_2 - a_1^2}}\right) \arctan\left(\frac{a_1 + 2a_2 p_i}{\sqrt{4a_0a_2 - a_1^2}}\right),$$

from which:

$$f(p_i) = \left[\frac{\mathcal{N}}{(a_0 + a_1 p_i + a_2 p_i^2)^{1+z_1/a_2}}\right] e^{4\left(\frac{z_0 + z_1 a_1/(2a_2)}{\sqrt{4a_0a_2 - a_1^2}}\right) \arctan\left(\frac{a_1 + 2a_2 p_i}{\sqrt{4a_0a_2 - a_1^2}}\right)},$$

where $\mathcal{N}$ is such that $\int_{-\infty}^{\infty} f(p_i)\, dp_i = 1$. Finally we notice that the distribution $f(p_i)$ is well-defined if $\sqrt{4a_0a_2 - a_1^2}$ has real roots, that is $4a_0a_2 - a_1^2 > 0$. Since:

$$4a_0a_2 - a_1^2 = 4\left[\Delta s^2 (1-\tau_l)(1-\tau_k)\rho^* \omega \bar{l}\right]^2 \left(\bar{\Phi}\bar{\Theta} - \bar{\Omega}^2\right)$$

and therefore:

$$4a_0a_2 - a_1^2 > 0 \Leftrightarrow \bar{\Phi}\bar{\Theta} - \bar{\Omega}^2 > 0;$$

taken dynasty i, from Eq. (20) we have

$$\sum_{j=1}^{F} \phi_{i,j}^2 \left(\sum_{j=1}^{F} \theta_{i,j}^2\right) > \left(\sum_{j=1}^{F} \theta_{i,j}\phi_{i,j}\right)^2,$$

which holds true, by the Cauchy inequality (see Hardy et al. (1954)).

QED

E Proof of Proposition 9

The proof follows the same steps of proof of Proposition 2 in Appendix D. When $N \to \infty$ from Eqq. (29) and (58) we have:

$$\mu(p_i) = z_0 - z_1 p_i \text{ and} \tag{85}$$

$$\sigma^2(p_i) = a_2 p_i^2, \tag{86}$$

where:

$$z_0 = s\left[\omega^* + \tau_k \rho^* \bar{p}\right] - \chi;$$

$$z_1 = \nu - s(1-\tau_k)\rho^* \text{ and}$$

$$a_2 = \Delta s^2 (1-\tau_k)^2 \rho^{*2} \bar{\Theta}.$$

The marginal distribution satisfies:

$$f^{SW}(p_i) = \left[\frac{B}{\sigma^2(p_i)}\right] e^{2\int_{-\infty}^{p_i} dx_i \mu(x_i)/\sigma^2(x_i)}, \tag{87}$$

where B is a constant. Therefore:

$$f^{SW}(p_i) = \frac{\mathcal{N}^{SW}}{a_2 p_i^{2(1+z_1/a_2)}} e^{-\left(\frac{z_0}{a_2 p_i}\right)},$$

where $\mathcal{N}^{SW}$ is such that $\int_{-\infty}^{\infty} f^{SW}(p_i)\, dp_i = 1$.

QED

F Proof of Proposition 10

Condition (60) ensures that $\bar{p}$ can grow forever, in fact from Eq. (39) we have that for large value of $\bar{p}$ $d\bar{p}/dt > 0\ \forall \bar{p} > 0$. If also Condition (61) holds then $d\bar{p}/dt > 0$ for $\bar{p} = 0$. Therefore for the concavity of $g(\cdot)$ $d\bar{p}/dt > 0$ for all $\bar{p} \geq 0$. This case is reported in Figure 4. Otherwise if Condition (63) holds, then $d\bar{p}/dt < 0$ for $\bar{p} = 0$. Since $\lim_{\bar{p}\to\infty} d\bar{p}/dt > 0$ and $g(\cdot)$ is concave, then there exists one value of $\bar{p}$, $\bar{p}^*$, such that $d\bar{p}/dt = 0$ and $d\bar{p}/dt > 0$ for all $\bar{p} > \bar{p}^*$ (see Figure 5). The economy will be therefore growing in the long run if the initial value of per capita wealth is higher than $\bar{p}^*$, otherwise $\bar{p}$ converges towards zero. Finally, Eqq. (64) and (65) are directly derived by Eqq. (15) and (16), taking into account Eqq. (18) and (19), given that $\lim_{\bar{p}\to\infty} g(\bar{p})/\bar{p} = \lim_{\bar{p}\to\infty} g'(\bar{p})$.

QED

G Proof of Proposition 11

Given the definition of u_i we have that:

$$\frac{du_i}{dt} = \frac{dp_i}{dt}/\bar{p} - u_i \frac{d\bar{p}}{dt}/\bar{p}.$$

From Eqq. (40) and (64) we have that:

$$\lim_{t\to\infty} \frac{d\bar{p}}{dt}/\bar{p} = s\rho^* - v,$$

given that $\lim_{\bar{p}\to\infty} g(\bar{p})/\bar{p} = \lim_{\bar{p}\to\infty} g'(\bar{p})$. Moreover, from Eq. (29) we have that:

$$\lim_{t\to\infty} \frac{dp_i}{dt}/\bar{p} = s\left[(1-\tau_k)\rho^* u_i + \tau_k \rho^*\right] - \nu u_i + \eta_i/\bar{p},$$

given that $\lim_{\bar{p}\to\infty}\omega^* = 0$ (see Eq. (65), taking into account that $\lim_{\bar{p}\to\infty} g(\bar{p})/\bar{p} = \lim_{\bar{p}\to\infty} g'(\bar{p})$). Therefore:

$$\lim_{t\to\infty}\frac{du_i}{dt} = s\left(1-\tau_k\right)\rho^* + \tilde{\eta}_i,$$

where $\tilde{\eta}_i = \eta_i/\bar{p}$. The derivation of $\lim_{t\to\infty}\lim_{N\to\infty} H_{i,i'}\left[\vec{u}\right]$ follows the same steps reported in Proposition 1. In fact:

$$E\left[\tilde{\eta}_i\left(t\right)\tilde{\eta}_{i'}\left(t'\right)\right] = \frac{1}{\bar{p}\left(t\right)\bar{p}\left(t'\right)}E\left[\eta_i\left(t\right)\eta_{i'}\left(t'\right)\right] = \frac{H_{i,i'}\left[\vec{p}\right]}{\bar{p}\left(t\right)\bar{p}\left(t'\right)}\delta\left(t-t'\right),$$

where $H_{i,i'}\left[\vec{p}\right]$ is the same as in Proposition 1, taken into account that in the limit $\bar{p}\to\infty$ only the quadratic terms in $p_i = \bar{p}u_i$ survives.

QED