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THE INFLUENCE OF THE CAPITAL-OUTPUT RATIO ON REAL NATIONAL INCOME[1]

BY MAURICE ALLAIS

This paper presents first a general theory of a capitalistic optimum and a model illustrating its essential features, and, secondly, the empirical justification of this model, and its principal applications.

Under very general conditions it is possible to show that we cannot expect, from an indefinite increase of available real capital, an indefinite increase of real national income consumed per inhabitant, and that there is an optimum amount of capital for which the real income per inhabitant is maximum. The conditions under which this maximum is attained are given.

The general model, which is presented, and, in particular, its exponential variety, appear quite remarkably confirmed by all presently available empirical data, with respect to both the hypotheses and the results.

A *very simple* expression of *consumed* real income is given in terms of the rate of interest i and the rate of growth ϱ.

INTRODUCTION

THE OBJECT of this paper is to explain the essential part of the results obtained in the research I have carried on at various times in the past twenty years[2] on the theory of a capitalistic optimum. Till now this theory has received, in my opinion, insufficient attention in the economics literature.

In the limited space available I intend to present, first, a general theory of a capitalistic optimum and a model illustrating its essential features, and, secondly, the empirical justification of this model, and its principal applications; for it is not a good theory if there is no verification of its results and hypotheses by facts, and if no fruitful application can be derived.

Essentially, I intend to show that we cannot expect, from an indefinite increase of available real capital, an indefinite increase of real national income consumed per capita, and that there is an optimum amount of capital for which real income per capita is a maximum.

[1] This article is adapted from the Bowley-Walras Lecture delivered by Professor Allais at the American Meetings of the Econometric Society, December 28, 1961. This invited lecture was the first in a series made possible by funds provided to the Econometric Society by the U.S. National Science Foundation.—ed.

[2] In 1940, 1946, 1954, 1960 and 1961. A large part of my results have already been published in 1947 in my book *Economie et Intérêt*, and in 1960 in my communication to the Congress of the International Statistical Institute at Tokyo. Moreover, the whole of the results I have obtained will be published very soon in English as a volume in the series "Contributions to Economic Analysis," North-Holland Publishing Co. The indications that follow are naturally not sufficient by themselves and they constitute only a memorandum.

Curiously enough, for a century now very few works have really been concerned with the influence of capital on real income, either on empirical or theoretical grounds, although such research is of the greatest importance from both the theoretical and practical points of view. *At the empirical level,* most research gives the idea that real national income can be increased indefinitely by using more and more capital. *At the theoretical level,* many works suggest the same conclusion, and though the idea of a capitalistic optimum for a zero rate of interest was positively stated by Wicksell as early as 1901, it was not until Meade's *Economic Analysis and Policy* appeared in 1937 that a more systematic, though literary statement was given. For my part, I have met in this area two sorts of economists. For the first, the existence of a capitalistic optimum for a zero rate of interest is considered as a completely mistaken proposition; for the second, it appears as a commonplace truth, a sort of truism, that does not deserve any serious attention at all.

1. GENERAL THEORY OF THE CAPITALISTIC OPTIMUM

1.1. In this section, I intend to show, at least briefly, that in fact there is a situation of a capitalistic optimum, and that such a situation is characterized, *under stationary conditions,* by a rate of interest equal to zero, and, *in a dynamic evolution,* in which the index of labour and natural wealth increases exponentially at a rate ϱ, by a rate of interest equal to ϱ.

A. *General Production Process*

1.2. *Definitions.* I define the input and output vectors of the production processes considered as follows:

(1.1)

outputs:

- $\vec{Q}_n$ is a representative vector of the production per unit of time of consumption goods.
- $\vec{E}_n$ is a representative vector of the production per unit of time of the various forms of equipment.

inputs:

- $\vec{X}_n$ is a representative vector of labour services and natural resources consumed per unit of time (primary production factors).
- $T\vec{E}_{n-p}$ is a representative vector of equipment of age p used during the period T_n and produced during the period T_{n-p}.

The indexes, Q_n and X_n, of the production of consumption goods and of the consumption of primary factors, respectively, are defined by:

$$Q_n = Q\,[\vec{Q}_n]\,, \tag{1.2}$$

$$X_n = X[\vec{X}_n]\,. \tag{1.3}$$

Each of these indexes is an increasing function of each coordinate of its argument.

1.3. *Hypotheses.* The production process is assumed to satisfy four hypotheses (a_1)–(a_4).

HYPOTHESIS a_1: *Pareto optimality over time.* We consider a production process, P, over time, and we suppose that between two extreme situations, corresponding to any two instants, a Pareto optimum is realized in the sense that no production could be increased at any intermediate time without production being diminished at another time.[3] It is then possible to write

$$f_n(\vec{Q}_n, \vec{E}_n, \vec{X}_n, \vec{E}_{n-1}, \ldots, \vec{E}_{n-p}, \ldots) = 0\,. \tag{1.4}$$

HYPOTHESIS a_2: *Homogeneity of order k.*

$$f_n(\lambda^k\vec{Q}_n, \lambda\vec{E}_n, \lambda\vec{X}_n, \lambda\vec{E}_{n-1}, \ldots, \lambda\vec{E}_{n-p}, \ldots) = 0\,, \qquad k > 1\,, \tag{1.5}$$

for all λ.

HYPOTHESIS a_3:

$$|\vec{X}_n| < \varepsilon \text{ implies } |\vec{Q}_n + \vec{E}_n| < \varepsilon' \tag{1.6}$$

where ε' depends only on ε and tends to zero with ε. This says that the production of consumption and equipment goods is assumed to tend to zero when the vector of primary services of labour and natural resources tends to zero.

HYPOTHESIS a_4: *Hypothesis of advantageous growth.* Both indexes

$$Q_n^* = Q\left[\frac{\vec{Q}_n}{X_n^k}\right], \qquad X_n \tag{1.7}$$

are never decreasing.

The index Q_n^*, here defined, represents, for an order of homogeneity equal to one, the production of primary factors per unit of consumption.

1.4. We now state a general theorem about the evolution of an advantageous capitalistic process:

[3] Allais [1943, Chap. VII, pp. 604–682] and [1947A, Chap. VI, pp. 153–178].

THEOREM I:[4] *If hypotheses* (a_1) *to* (a_4) *are satisfied then:*

(a) *The index* Q_n^* *tends, under very general conditions, to an upper limit when* n *increases indefinitely.*

(b) *Under equally general conditions, the process* P', *given by*

$$\vec{Q}'_n = \frac{\vec{Q}_n}{X_n^k}, \qquad \vec{X}'_n = \frac{\vec{X}_n}{X_n}, \qquad \vec{E}'_n = \frac{\vec{E}_n}{X_n}, \tag{1.8}$$

tends to an asymptotic process, P'_a, *of stationary equilibrium* $[\vec{Q}', \vec{X}', \vec{E}']$ *in which the rate of interest has a* constant *value* i'. *The process* P *tends to an asymptotic process* P_a:

$$\vec{Q}_n^a = X_n^k \vec{Q}', \qquad \vec{X}_n^a = X_n \vec{X}', \qquad \vec{E}_n^a = X_n \vec{E}'. \tag{1.9}$$

(c) *The process* P_a *is characterized by a rate of interest*

$$i = \varrho + i' \tag{1.10}$$

with

$$\varrho = \frac{1}{X}\frac{dX}{dt}, \tag{1.11}$$

ϱ *being the rate of growth of primary income, and* i' *being the rate of interest characterizing the stationary process* P'_a.

In process P_a *all relative prices are constant.*

This theorem, *which can easily be extended to convex production functions*, has, in particular, the advantage of replacing the study of dynamic processes by the study of stationary ones.

B. *General Economic Definitions*

1.5. In this subsection, I give some general definitions and relations among them. In doing so, it has seemed better to me to keep the French notation corresponding to my previous work.

Definitions:

		In labor units[7]	In real units
Capital	Reproducible national capital[5]	C	$\bar{C}$
	Land capital[6]	C_φ	
	Total national capital	C_T	

[4] This theorem may be easily extended to the case in which the production function (1.4) is convex.

[5] Value of natural resources excluded.

[6] Capitalized value of natural resources.

[7] Nominal values divided by the nominal value of one hour of unskilled labor.

Income	National income	R	$\bar{R}$
	Consumed national income	R_c	$\bar{R}_c$
	Primary income[8]	R_ω	
	from labor	R_σ	
	from land	R_φ	

Relations:

$$\begin{cases} R = R_c + \dfrac{dC}{dt}, & (1.12) \\ R = R_\omega + iC, & (1.13) \\ R_\omega = R_\sigma + R_\varphi, & (1.14) \\ R_\varphi = iC_\varphi, & (1.15) \\ C_T = C + C_\varphi, & \end{cases}$$

(1.16) Capitalistic

(1.17) Characteristics

i, the instantaneous rate of interest;

$\varrho = \dfrac{1}{R_\omega}\dfrac{dR_\omega}{dt}$, the rate of growth of primary income;

$\gamma = C/R$, the capital-output ratio defined by reference to the income R;

$\gamma_c = C/R_c$, the capital-output ratio defined by reference to the consumed income R_c.

C. *Characteristic Functions*

1.6. I now define the concept of the characteristic function, which to my knowledge was presented for the first time by Jevons in 1871, in connection with a particular model. Here is the simplest way to conceive of the characteristic curve of a production process, the curve which is represented in Figure 1. Let us suppose that we pay wages for the construction of a blast furnace. This blast furnace will produce cast iron, which will be used to produce steel. The steel will be sent to manufacturing industries, and finally automobiles can be made out of it; and these automobiles the consumers can buy for their personal use. The expenditure on wages for the construction of the blast furnace will finally become part of the sales price of the automobiles. It will therefore appear in the national income after a certain delay.

We see then that the national income consumed at a certain time includes expenditures on wages made at an earlier time, and we can construct a curve, at least at the theoretical level, that gives the distance in time of the various

[8] Value of primary factors of production (wages and rents).

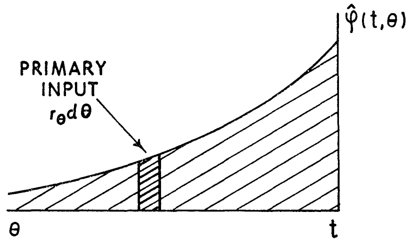

FIGURE 1.—Average period of production.

$$\hat{\Theta}(t) = \int_0^\infty \theta\hat{\varphi}(t, \theta)\, d\theta$$

previous expenditures for labour and land services which appear in the national income consumed at a given moment. This we call the "characteristic curve"[9] of the capitalistic process being studied. *By definition the primary income is imputed proportionally to the marginal productivities in physical values.*[10]

Further definitions are given in the following tableau:

		supplied at	appearing in the income consumed at	Primary inputs
Primary income	elementary	t	$t + \theta$	$r_\theta d\theta = R_\omega(t)\,\varphi(t, \theta)\, d\theta$
		$t - \theta$	t	$r_\theta d\theta = \hat{R}_\omega(t)\hat{\varphi}(t, \theta)\, d\theta$
	total	t		$R_\omega = R_\omega \int_0^\infty \varphi(t, \theta)\, d\theta$
			t	$\hat{R}_\omega = \hat{R}_\omega \int_0^\infty \hat{\varphi}(t, \theta)\, d\theta$

In the same way, we can define an amortization curve, represented in Figure 2, which gives the distribution of the primary income of time t among the different periods that follow.

Under stationary conditions, the curves of Figures 1 and 2 are, of course, exactly symmetrical.

The average production period

$$\Theta(t) = \int_0^\infty \theta\hat{\varphi}(t, \theta)\, d\theta\,, \tag{1.18}$$

and the average amortization period is

$$\Theta = \int_0^\infty \theta\varphi(t, \theta)\, d\theta\,. \tag{1.19}$$

[9] Naturally, this concept has nothing in common with the concept of the same denomination used in statistics.

[10] See Allais, *Economie et Intérêt,* p. 118, note (2′) and *Traité d'Economie Pure,* no. 312.

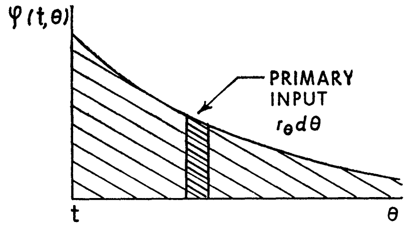

FIGURE 2.—Average period of amortization.

$$\Theta = \int_0^\infty \theta\varphi(t, \theta)\, d\theta$$

Consumed income is

$$(1.20)\quad R_c = \int_0^\infty r_\theta e^{i\theta} = \int_0^\infty \hat{R}_\omega(t)\hat{\varphi}(t, \theta)\, e^{i\theta} d\theta = \int_0^\infty R_\omega(t-\theta)\,\varphi(t-\theta,\theta)\, e^{i\theta} d\theta\,,$$

and the value of the capital stock is

$$(1.21)\qquad C = \int_0^\infty dt \int_T^\infty R_\omega(t+T-\theta)\,\varphi(\theta)\, e^{i(\theta-T)}\, d\theta \quad \text{for } \varphi \text{ independent of } t.$$

For the particularly simple case of stationary conditions, we have

$$(1.22)\qquad \hat{R}_\omega = R_\omega\,, \qquad \hat{\varphi} = \varphi\,,$$

these functions being independent of t. Hence,[11]

$$(1.23)\qquad R = R_c = R_\omega \int_0^\infty e^{i\theta}\varphi(\theta, i)\, d\theta\,,$$

$$(1.24)\qquad C = R_\omega \int_0^\infty \frac{e^{i\theta}-1}{i}\,\varphi(\theta,i)\, d\theta\,.$$

D. *Capitalistic Optimum in a Stationary Process*

1.7. *Definition of a capitalistic optimum in a stationary process.* In a stationary process, equation (1.4) can be written

$$(1.25)\qquad g[\vec{Q}, \vec{X}, \vec{E}] = 0\,.$$

By definition, there is a capitalistic optimum if for a given vector $\vec{X}$ we have for any coordinate Q_i of vector $\vec{Q}$

$$(1.26)\qquad \delta Q_i \leqslant 0 \qquad \text{whatever be } \delta\vec{E}\,,$$

the other coordinates of $\vec{Q}$ being maintained constant.[12] This is to define a capitalistic optimum in a stationary process by the impossibility of increasing the production of consumption goods by any variation of the equipment vector.

[11] See Allais [1947A, pp. 118–131].

[12] See Allais [1947A, pp. 180–181].

Figure 3 shows how the surface of maximal possibilities varies when the rate of interest varies.

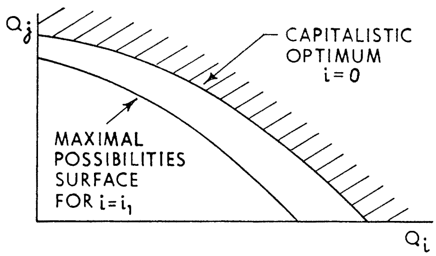

FIGURE 3

1.8. *Simplified analysis by periods.* We next present the theory of a capitalistic optimum for the very simple case of a period analysis with a single kind of labour and with a given production function that determines real income as a function of the primary inputs, and that is homogeneous of degree one. (The characteristic curve is represented in Figure 4.)

Let X_p be the amount of labour of period T_p contributing to the production of the instant t.

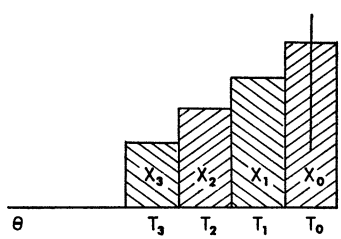

FIGURE 4

Let

(1.27) $$\bar{R} = f(X_0, X_1, \ldots, X_n, \ldots)$$

be the production function and be homogeneous of order one. The elasticities are $d\bar{R}/\bar{R} \div dX_n/X_n$.

At equilibrium the market value of production is

(1.28) $$R = r\bar{R} = xX_0 + e^{iT}xX_1 + \ldots + e^{inT}xX_n + \ldots,$$

where x is the wage rate, i is the instantaneous rate of interest in wage units, and r is price per unit of output.

The functioning of the market leads to a maximization of profit subject to constraint (1.27), the prices x and i being considered as constant. Thus we have

(1.29) $$\frac{1}{r} = \frac{\dfrac{\partial\bar{R}}{\partial X_0}}{x} = \frac{\dfrac{\partial\bar{R}}{\partial X_1}}{xe^{iT}} = \ldots = \frac{\dfrac{\partial\bar{R}}{\partial X_n}}{xe^{inT}} = \ldots.$$

Expressing the conditions for a maximum of $\bar{R}$ subject to the constraint

$$X_0 + X_1 + \ldots + X_n + \ldots = X \tag{1.30}$$

we obtain

$$\frac{\partial \bar{R}}{\partial X_0} = \frac{\partial \bar{R}}{\partial X_1} = \ldots = \frac{\partial \bar{R}}{\partial X_n} = \ldots . \tag{1.31}$$

From (1.29) and (1.31) we have

$$i = 0 \tag{1.32}$$

as the condition for a capitalistic optimum, *considering as given the total amount of labour available at any time.*

1.9. *Analysis for the general case.*[13] In this case national income is given by

$$R = R_c = R_\omega \int_0^\infty e^{i\theta} \varphi(\theta, i)\, d\theta \, . \tag{1.33}$$

Under the assumption of homogeneity of order k, we obtain, according to the theory of the optimum allocation of resources, the index $\bar{R}$ of real income by differentiating, *at constant prices,* expression (1.33) for national income. Since the prices of the primary factors are represented here by the values of $e^{i\theta}$, this differentiation at constant prices leads to

$$\frac{1}{k}\,\frac{1}{\bar{R}}\,\delta\bar{R} = \frac{\int_0^\infty e^{i\theta}\,\delta\varphi(\theta, i)\, d\theta}{\int_0^\infty e^{i\theta}\,\varphi(\theta, i)\, d\theta}\,, \qquad \frac{d\Theta}{di} < 0\,. \tag{1.34}$$

In the vicinity of $i = 0$, we have

$$\frac{1}{k}\,\frac{1}{\bar{R}}\,\frac{\partial \bar{R}}{\partial i} \backsim i \int_0^\infty \theta\, \frac{\partial \varphi(\theta, i)}{\partial i}\, d\theta = i\, \frac{d\Theta}{di}\,, \tag{1.35}$$

$$\frac{1}{k}\,\frac{\bar{R}_M - \bar{R}}{\bar{R}_M} \backsim -\frac{i^2}{2}\,\frac{d\Theta}{di}\,, \tag{1.36}$$

$$\gamma = C/R \backsim \Theta\,, \tag{1.37}$$

$$R \backsim R_\omega(1 + i\Theta)\,. \tag{1.38}$$

Equations (1.35) and (1.37) give the expressions for the index $\bar{R}$ and the capital-output ratio γ, in the vicinity of $i = 0$. The first one shows that real national income is stationary for a zero value of the rate of interest. The second one shows that for small values of the rate of interest, the capital-output ratio γ is, in a stationary process, approximately equal to the average production period Θ. We therefore state

[13] Allais [1947A, pp. 186–189 and 194–206].

THEOREM II (*Condition for a capitalistic optimum under stationary conditions*): *Real income consumed is a maximum for* $i = 0$.

E. *Capitalistic Optimum of a Dynamic Capitalistic Process*

1.10. The combined results of Theorems I and II lead to the following theorem concerning dynamic processes.

THEOREM III (*Condition for a capitalistic optimum in a dynamic process*): *Using the earlier notation, since we have*

$$\frac{\vec{Q}_n}{X^k_n} \to \vec{Q}'$$

and also that $Q(\vec{Q}')$ *is maximum for* $i' = 0$, *we may infer that*

$$Q^*_n = Q\left[\frac{\vec{Q}_n}{X^k_n}\right]$$

is asymptotically maximum for

$$i' = i - \varrho = 0\,.$$

Thus, among all processes P_a *that are characterized by a rate of interest* $i = \varrho + i'$, *the one for which the consumed real income is highest, at any instant, corresponds to the rate of interest* $i = \varrho$.

2. THE MODEL

2.1. The general theory, the most important aspects of which I have just sketched, may be illustrated by a very simple model, which appears to have great generality. I do not claim that this model should be "the" model to illustrate in a definitive way the theory of capital. I only think that this model is very simple and highly suggestive. It has, among other advantages, the essential one of lending itself easily to numerical applications, and, in particular, of allowing an easy estimate of the influence of utilized capital on consumed real national income.

A. *Hypotheses*

2.2. *General Model.* This general model rests on the hypotheses (a_1) to (a_4) of Section 1[14] and on

HYPOTHESIS *b*: *Constancy of the production elasticities.* At any instant *t*, the elasticities of the consumed real income with respect to the primary inputs may be considered as constant in a large region and independent of the instant *t*.

[14] Here we put:

$$Q(t) = \bar{R}_c(t), \qquad X(t) = R_\omega(t)\,.$$

This hypothesis amounts to supposing that, in a very large region,

$$(2.1) \qquad L\bar{R}_c(t) = L\alpha + \int_0^\infty \beta(\theta) L\hat{r}_\theta \, d\theta \, ,$$

that is to say that the total production function is logarithmically linear and invariant with time.

Notation. The Laplace transform ψ of the function β, by which all results can be simply written, is

$$(2.2) \qquad \psi(u) = \frac{1}{k} \int_0^\infty \beta(\theta) \, e^{-\mu\theta} du \, .$$

In addition, we define the three fundamental constants of the general model, k, Θ_0 and Δ, the consideration of which is sufficient in all empirical applications, by limiting ourselves to the first three terms of the Taylor series expansion of the function ψ:

$$(2.3) \quad \left\{ \begin{aligned} k &= \int_0^\infty \beta(\theta) \, d\theta \\ \Theta_0 &= \frac{1}{k} \int_0^\infty \theta \beta(\theta) \, d\theta \\ \Delta &= \frac{1}{2k\Theta_0^2} \int_0^\infty \theta^2 \beta(\theta) \, d\theta \end{aligned} \right. \qquad \left\{ \begin{aligned} &\psi(u) \backsim 1 - \Theta_0 u \\ &\frac{d\psi}{du}(u) \backsim -\Theta_0[1 - 2\Delta\Theta_0 u] \\ &\Delta > \tfrac{1}{2} \text{ (in any case)} . \end{aligned} \right. \quad \left\{ \begin{aligned} &\text{for } u \\ &\text{small,} \end{aligned} \right.$$

2.3. *Exponential Model.* A particular variety of the general model, which I call the "Exponential Model," requires, moreover:

HYPOTHESIS *c*: *Exponential decrease of the elasticities of production.*

$$(2.4) \qquad \beta(\theta) = \frac{k}{\Theta_0} e^{-\frac{\theta}{\Theta_0}} .$$

This hypothesis, which assumes an exponential decrease of production elasticities β over time, does not seem unrealistic.

B. *Consequences of Hypotheses*

2.4. *General Model. Consequences of Hypotheses* (*a*) *and* (*b*). We next cite the principal consequences of hypotheses (a_1) to (a_4) and (b). The capitalistic process P (shown on the left of the next page) tends to the asymptotic process P_a (shown on the right) with:

2.5–2.5*	$i(t) - \varrho(t) = i'$ (independent of t)	$\varrho(t) = \dfrac{1}{R_\omega}\dfrac{dR_\omega(t)}{dt}$	
2.6–2.6*	$\left\{\begin{array}{l} \varphi(t,\theta) = \dfrac{\beta(\theta)e^{-(i-\varrho)\theta}}{k\psi(i-\varrho)} \\ \hat{\varphi}(t,\theta) = \beta(\theta)\,\dfrac{e^{-i\theta}}{k\psi(i)} \end{array}\right.$	$\psi(u) \backsim 1 - \Theta_0 u + \Delta\Theta_0^2 u^2$ $\hat{R}_\omega(t) = \dfrac{\psi(i)}{\psi(i-\varrho)}\,R_\omega(t)$	[16] [15]
2.7–2.7*	$\Theta(i = \varrho) = \Theta_0$ $\Theta(i = 0) = \Theta_0$	$\left\{\begin{array}{l} \Theta \backsim \Theta_0[1 - 2\Theta_0(\Delta - \frac{1}{2})(i-\varrho)] \\ \Theta \backsim \Theta_0[1 - 2\Theta_0(\Delta - \frac{1}{2})i] \end{array}\right.$	[16]
2.8–2.8*	$\left\{\begin{array}{l} R_c(t)/R_\omega(t) = 1/\psi(i-\varrho) \\ R(t)/R_\omega(t) = \dfrac{i - \varrho\psi(i-\varrho)}{(i-\varrho)\psi(i-\varrho)} \end{array}\right.$	$R_c/R_\omega \backsim 1 + \Theta_0(i-\varrho)$ $R/R_\omega \backsim 1 + \Theta_0 i$	[16] [16]
2.9–2.9*	$C(t)/R_\omega(t) = \dfrac{1}{(i-\varrho)}\left[\dfrac{1}{\psi(i-\varrho)} - 1\right]$	$C/R_\omega \backsim \Theta_0[1 + \Theta_0(1-\Delta)(i-\varrho)]$	[16]
2.10–2.10*	$\left\{\begin{array}{l} \gamma_c = \gamma_c(i,\varrho) = C(t)/R_c(t) = \dfrac{1-\psi(i-\varrho)}{i-\varrho} \\ \gamma_{c,0} = \gamma_c(i=\varrho) = \Theta_0 \end{array}\right.$	$\gamma_c \backsim \Theta_0[1 - \Delta\Theta_0(i-\varrho)]$	[16]
2.11–2.11*	$\left\{\begin{array}{l} \gamma = \gamma(i,\varrho) = C(t)/R(t) = \dfrac{1-\psi(i-\varrho)}{i-\varrho\psi(i-\varrho)} \\ \gamma_0 = \gamma(i=\varrho) = \dfrac{\Theta_0}{1+\varrho\Theta_0} \end{array}\right.$	$\gamma \backsim \Theta_0[1 - \Delta\Theta_0 i + \Theta_0(\Delta - 1)\varrho]$ $\Theta_0 \backsim \sqrt{\dfrac{\gamma_c - \gamma}{\varrho}} \qquad \Delta \backsim \left(1 - \dfrac{\gamma_c}{\Theta_0}\right)\Big/\Theta_0(i-\varrho)$	[16]
2.12–2.12*	$\dfrac{\bar{R}_c(t)}{\bar{R}_{cM}(t)} = \left[\dfrac{e^{-\Theta_0(i-\varrho)}}{\psi(i-\varrho)}\right]^k = \left[\dfrac{1-\gamma\varrho}{1-\gamma i}\,e^{-\Theta_0(i-\varrho)}\right]^k \backsim \left[1 - \Theta_0^2(\Delta - \frac{1}{2})(i-\varrho)^2\right]^k$	$\dfrac{\bar{R}_c}{\bar{R}_{cM}} = \left[\dfrac{R_c}{R_\omega}\,e^{-\Theta_0(i-\varrho)}\right]^k$	
2.13	$\dfrac{\bar{R}(t)}{K(t)} = \left[\dfrac{i-\varrho\psi(i-\varrho)}{(i-\varrho)\psi(i-\varrho)}\,e^{-\Theta_0 i}\right]^k = \left[\dfrac{1+\varrho\gamma_c}{1-(i-\varrho)\gamma_c}\,e^{-\Theta_0 i}\right]^k \backsim \left[1 - \Theta_0^2(\Delta - \frac{1}{2})i^2 + \Theta_0^2(\Delta - 1)\varrho i\right]^k$		[16, 17]
2.14–2.14*	$\dfrac{d\bar{R}_c}{di}(i = \varrho) = 0$	$\bar{R}_{cM} = K' e^{-k\varrho\Theta_0}$	[15]

[15] For ϱ independent of t. [16] For small ϱ and i. [17] We suppose $\bar{R}/\bar{R}_c = [R/R_c]^k$.

Derivations of these relations are based essentially on the fact that the distribution of primary inputs must maximize consumed income, in labor terms, under the constraint of the production function. I do not reproduce the derivations here.

The main results are as follows:

I must first point out the *very simple* expression for *consumed* real income given by equation (2.12) in terms of the rates i and ϱ. Relation (2.14) verifies that this real income actually reaches a maximum when the interest rate i is equal to the rate of growth ϱ of primary income. It is also interesting to note that for a constant value of ϱ *the maximum* consumed real income that can be reached at any time is represented by the relation (2.14*) which is a decreasing function of the expansion rate ϱ, so that the most advantageous stationary process is better than any process of advantageous growth. All these results are of course *independent of the particular form of the function* β, which makes them quite general.

2.5. *Exponential Model. Consequences of Hypotheses* (a),(b) *and* (c). Below I indicate the form that is taken by the preceding relations of the general model, when the function β decreases exponentially according to Hypothesis (c) above. The capitalistic process P (below on the left) tends to the asymptotic process (on the right) with:

2.15–2.15* $\quad i(t) - \varrho(t) = i'$ (independent of t) $\qquad \varrho(t) = \dfrac{1}{R_\omega}\dfrac{dR_\omega(t)}{dt}$ [18]

2.16–2.16*
$$\begin{cases} \varphi(t,\theta) = \dfrac{1}{\Theta}\, e^{-\frac{\theta}{\Theta}} & \qquad \Delta = 1, \qquad \psi(\mu) = \dfrac{1}{1+\Theta_0\mu} \\[2ex] \hat{\varphi}(t,\theta) = \dfrac{1}{\hat{\Theta}}\, e^{-\frac{\theta}{\hat{\Theta}}} & \qquad \hat{R}_\omega(t) = \left(1 - \dfrac{\Theta_0\varrho}{1+\Theta_0 i}\right)R_\omega(t) \end{cases}$$

2.17–2.17*
$$\begin{cases} \Theta = \Theta_0/[1+\Theta_0(i-\varrho)] \\ \hat{\Theta} = \Theta_0/[1+\Theta_0 i] \end{cases} \qquad \frac{1}{\hat{\Theta}} - \frac{1}{\Theta} = \varrho$$

2.18
$$\begin{cases} R_c = [1+\Theta_0(i-\varrho)]R_\omega \\ R = [1+\Theta_0 i]R_\omega \end{cases}$$

2.19 $\quad C(t) = \Theta_0 R_\omega(t)$ [18]

2.20–2.20*
$$\begin{cases} \gamma_c = C(t)/R_c(t) = \Theta \\ \gamma_{c,0} = \gamma_c(i=\varrho) = \Theta_0 \end{cases} \qquad \Theta = \frac{\gamma}{1-\varrho\gamma}$$

2.21–2.21*
$$\begin{cases} \gamma = C(t)/R(t) = \hat{\Theta} \\ \gamma_0 = \gamma(i=\varrho) = \dfrac{\Theta_0}{1+\varrho\Theta_0} \end{cases} \qquad \Theta_0 = \frac{\gamma}{1-i\gamma}$$

[18] For ϱ independent of t.

2.22–2.22* $\frac{\bar{R}_c}{\bar{R}_{cM}} = \left[\frac{\Theta_0}{\Theta} e^{-\frac{\Theta_0}{\Theta}}\right]^k$ $\frac{\bar{R}_c}{\bar{R}_{cM}} = \{[1 + \Theta_0(i - \varrho)] e^{-\Theta_0(i-\varrho)}\}^k$

2.23–2.23* $\frac{\bar{R}}{\bar{R}_M} = \left[\frac{\Theta_0}{\Theta} e^{1-\frac{\Theta_0}{\hat{\Theta}}}\right]^k$ $\frac{\bar{R}}{\bar{R}_M} = [(1 + \Theta_0 i) e^{-\Theta_0 i}]^k$ [19]

(2.24–2.24* –2.24**) $\frac{d\bar{R}_c}{di}(i = \varrho) = 0$ $\bar{R}_{cM} = K' e^{-k\varrho\Theta_0}$ [18] $\frac{d\bar{R}}{di}(i = 0) = 0.$

Here, all useful variables can be expressed in a remarkably simple manner; the relations obtained depend only on two constants: the time constant Θ_0, and the degree of homogeneity k. The basic constant Θ_0 is very simply expressed by relation (2.21*) in terms of two observed variables: the capital-output ratio, γ, and interest rate, i, both observable data.

It remains true, of course, that consumed real income, given by relation (2.22), *reaches a maximum when the interest rate i is equal to the growth rate ϱ of primary income* (*relation* 2.24).

In this case, it is also noteworthy that real national income, given by relation (2.23), reaches a maximum when the interest rate is equal to zero (relation (2.24**)), but it is possible to show that this property is not true when the function β has not the exponential form.

Finally, let me point out the very remarkable character of relation (2.19), according to which *the capital value is independent of the rate of interest* and equal to the product of the primary income R_ω by the constant Θ_0. As we shall see, *this quite surprising result can be empirically verified.*

C. *Generalisation of the Model*

2.6. The preceding model may be easily modified to take account of neutral technical progress on the one hand, and uncertainty about the future on the other.

If we rewrite the production function (2.1) as [19']

$$L\bar{R}_c(t) = L\alpha' + \int_0^\infty \beta_\theta L\left(\frac{\hat{r}_\theta}{e^{\pi\theta}}\right) d\theta\,,$$

with $\alpha' = \alpha_0 e^{\pi t}$, nothing changes, and so we can admit the possibility for α in (2.1) that it be given by

$$L\alpha = L\alpha' - \int_0^\infty \pi\theta\beta(\theta)\,d\theta\,.$$

[19] We suppose $\bar{R}/\bar{R}_c = [R/R_c]^k$.

[19'] Since one hour of work at time $t-\theta$ is equivalent to $e^{-\theta}$ hours of work at time t when the rate of technical progress is π.

TABLE I

VALUES OF THE CAPITAL-OUTPUT RATIO IN THE UNITED STATES FROM 1880 TO 1956

Years	C	R'	γ'	R	γ	i_n (in %)	σ (in %)	Θ_0 $i = i_n - \sigma$	Θ_0 $i = i_n$	Θ_0 Average
1880	24.7	7.7	3.20	7.9	3.13	6.47	1.5	3.71	3.93	3.82
1890	43.3	10.9	3.97	11.2	3.87	5.68	1.5	4.62	4.96	4.79
1900	59.1	15.7	3.76	16.0	3.69	5.18	1.5	4.27	4.56	4.42
Period 1880–1900 Averages			3.64		3.56	5.78	1.5	4.20	4.48	4.34
1906	83.6	24.3	3.44	24.8	3.37	5.56	1.5	3.90	4.15	4.03
1910	99.6	28.8	3.46	29.5	3.38	5.81	1.5	3.96	4.20	4.08
1913	113.8	32.2	3.53	33.1	3.44	6.06	1.5	4.08	4.34	4.21
Averages: Period 1906–1913			3.48		3.40	5.81	1.5	3.98	4.23	4.11
Averages: Period 1880–1913			3.56		3.48	5.80	1.5	4.09	4.36	4.23
1923	253.4	73.7	3.44	75.9	3.34	6.57	1.4	4.04	4.28	4.16
1929	313.2	87.8	3.57	90.5	3.46	6.48	1.4	4.20	4.46	4.33
1937	291.8	73.6	3.96	75.2	3.88	5.15	1.4	4.54	4.85	4.70
1923–1937 Averages			3.66		3.56	6.07	1.4	4.26	4.53	4.40
1950	836.9	241.9	3.46	246.1	3.40	3.86	3.95	3.39	3.91	3.65
1955	1110.6	330.2	3.36	336.3	3.30	4.25	3.95	3.33	3.84	3.59
1956	1199.6	350.8	3.42	357.8	3.35	4.57	3.95	3.42	3.96	3.69
Averages: Period 1950–1956			3.41		3.35	4.23	3.95	3.38	3.90	3.64
Averages: Period 1880–1956			3.54		3.46	5.47	2.09	3.95	4.29	4.12
Median 1880–1956			3.46		3.39	5.62	1.5	4.00	4.24	4.12
Average relative deviation			5%		5%	13%		8%	7%	8%

(1) The retained years were, as far as possible, full employment years.
(2) I have taken $i_n = i'_n + 1\%$ where i'_n represents the rate of interest of bonds.
(3) I have taken $R = R' + i_n C_e$ where C_e represents the value of durable consumption goods.
(4) $\gamma' = C/R'$; $\gamma = C/R$.
(5) We have $\Theta_0 = \gamma/(1 - i\gamma)$.
(6) The bottom row is the average in percentage of the deviations of the 12 values from their general average.
(7) Values of C, C_e and C_φ: Before 1896, from Goldsmith, "The Growth of Reproducible Wealth of the U.S.A. from 1805 to 1950," *Income and Wealth of the United States, Trends and Structure*, Series II, Bowes and Bowes, 1952, p. 306; after 1896, from National Balance Sheets and National Wealth Statements, 1896 to 1949," *A Study of Saving in the United States*, Princeton University Press, 1956, p. 14 (most of these values are given in the *Economic Almanac* of 1960, p. 389).
(8) Values of the national income R': for 1929 and from 1950 to 1956, *Economic Almanac of 1960*, National Income, p. 392; for 1900 to 1923 and 1937, *idem*, p. 410; for 1880 and 1890 *idem*, p. 398; the values 7.227 and 10.70 for 1879 and 1889 have been multiplied by 15.647/16.158 (ratio of the values for 1900) to make them comparable with the preceding values. To obtain the values of 1880 and 1890, I have, besides, multiplied the values by 1.037 to take into account the increased trend of the real value of the national income (Allais) and by 82/77 and 78/77 to take into account the general increase of prices.

Thus we find: $7.227 \times (15.647/16.158) \times (1.057 \times 82/77) = 7.7$;
$10.701 \times (15.647/16.158) \times (1.057 \times 78/77) = 10.9$.

(9) Values of i'_n: for 1950 to 1956, *Economic Almanac of 1960*, p. 80. Bond Yields (general average); before 1950, Macaulay, F. R., *Bond Yields, Interest Rates, Stock Prices*, National Bureau of Economic Research, 1938, Appendix A, 142, col. 5 (Adjust. Bonds). These values have been increased by 1% to make them comparable with the preceding ones (average correction for years 1925 and 1935).

Technical progress cannot therefore be considered as altering in a systematic way the production elasticities β.

We may also consider the function $\beta(\theta)$ as taking account of uncertainty about the future.

D. *Estimates of Model Parameters*

2.7. *The coefficient k of homogeneity*. The analysis of the statistical data of Rostas [1948], relating to the comparison of 31 American and English industries, leads to the conclusion that there is, on the whole, no substantial increasing returns to scale (the correlation coefficient between the logarithms of the ratios of productivities and number of employees is -0.43) (See Allais [1960A, Sec. 39; 1960B, pp. 36–39 and Appendix I-D, pp. 299–302]; and Allais [1961B, Appendix II].) This result seems corroborated by those of Douglas [1948], and is not opposed to those of Verdoorn [1949, 1950, 1956]. See Allais [1960A, Sec. 39] and [1960B, Appendix I-D].

All things considered, we can take the homogeneity coefficient to be slightly different from unity for the whole economy.

2.8. *The coefficient* Θ_0. Because of lack of space, the results obtained for the estimation of the constant Θ_0 are limited to the case of the exponential model, based on the relation (2.21*):

$$\Theta_0 = \gamma/(1 - i\gamma)\,, \qquad \gamma = C/R\,.$$

In Table I estimates are given for the U.S. for various (full employment) years and periods from 1880 to 1956 and in Table II estimates of Θ_0 are provided for the U.S., France, and Great Britain for 1913.

TABLE II

Capital-Output Ratio Values in 1913 in United States, France and Great Britain

	C	R'	γ'	R	γ	i_n (in %)	σ (in %)	Θ_0		
								$i = i_n - \sigma$	$i = i_n$	Average
United States	113.8	32.2	3.53	33.1	3.44	6.06	1.5	4.08	4.34	4.21
France	191.0	48.9	3.91	49.9	3.83	4.84	1	4.49	4.70	4.60
Great Britain (1914)	9138.5	2450	3.73	2490.4	3.67	5.00	0	4.50	4.50	4.50
Averages			3.72		3.64	5.30		4.36	4.51	4.44
Average relative deviation			3%		4%	12%		4%	3%	3%

Notes: $\gamma' = C/R'$, $R = R' + iC_e$, $\gamma = C/R$, $\Theta_0 = \gamma/(1 - i\gamma)$.
Sources: Allais, 1960A, *Influence du coefficient capitalistique sur le revenu national réel par tête* (Printed Text, n. 61, presented at the Tokyo Congress), pp. 52–53.

If we take into account the fact that in a real economy a dynamic equilibrium is only imperfectly realized, the rate of interest to be considered has an intermediate value between the nominal value i_n and this value minus σ, the rate of increase of nominal wages, without there being, in the present state of things, any possibility of going further. Thus I have indicated in

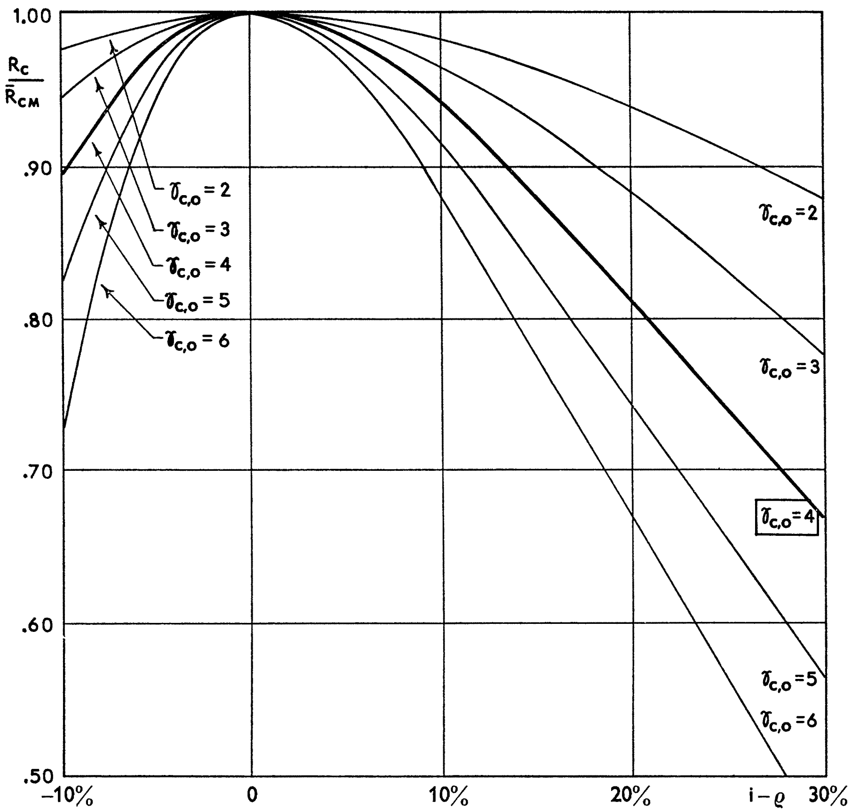

FIGURE 5.—Consumed real income as a function of the difference $i - \varrho$ in the case of a homogeneous production function for the exponential model.

$$\frac{\overline{R}_c}{\overline{R}_{cM}} = [1 + \gamma_{c,0}(i - \varrho)]\, e^{-\gamma_{c,0}(i-\varrho)}, \text{ equations (2.22*) and (2.20) ;}$$

$\overline{R}_c$ = real income consumed;
i = instantaneous rate of interest in labor units;
ϱ = rate of growth of primary income;
$\gamma_c = C/R_c = \Theta$, equation (2.20) ;
$\gamma_{c,0} = \gamma_c\,(i = \varrho) = \Theta_0$, equation (2.20);
$\gamma = C/R = \hat{\Theta}$, equation (2.21);
$\gamma_c = \dfrac{\gamma}{1 - \gamma_\varrho}$, equations (2.20) and (2.20*).

each case the two extreme estimates of Θ_0 to which we are led and the average.

The rate i_n is the rate of interest corresponding to a riskless loan, presenting no advantages of liquidity.

All the results, as shown in the tables, are in remarkable agreement, and show that the order of magnitude of the constant Θ_0 is 4.

In Figure 5, I indicate the shape of the curve representing consumed real income $\bar{R}_c$ as a function of the difference $(i - \varrho)$. We can see that we have a quite flat maximum for every possible value of Θ_0. Figure 6 represents consumed real income as a function of the capital-output ratio γ_c defined by reference to the consumed income R_c.

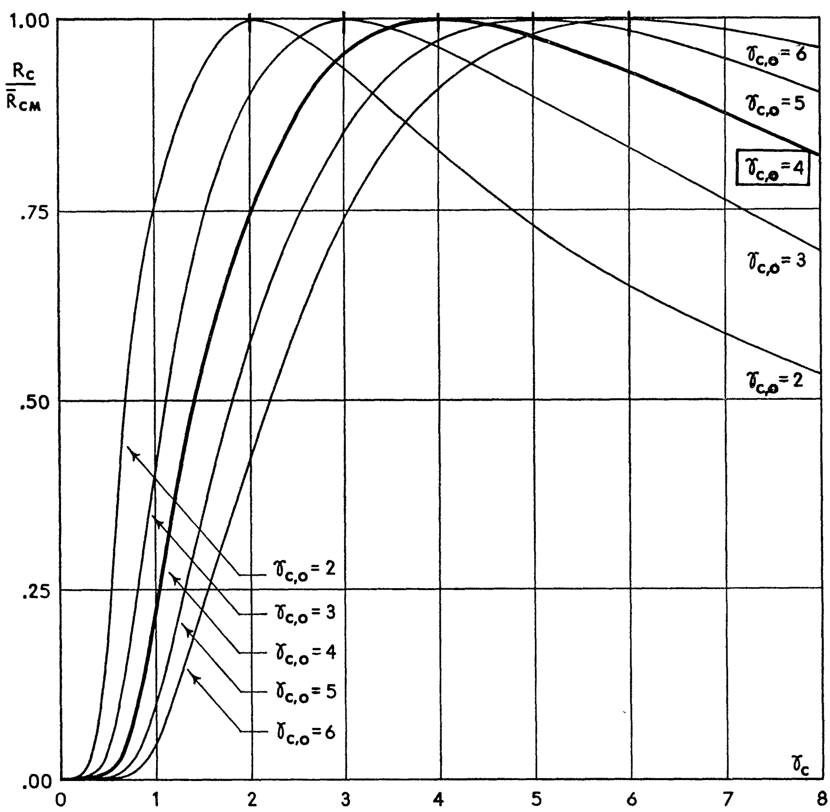

FIGURE 6.—Consumed real income as a function of the capital-output ratio in the case of a homogeneous production function for the exponential model.

$$\frac{\bar{R}_c}{\bar{R}_{cM}} = \frac{\gamma_{c,0}}{\gamma_c} e^{1-\frac{\gamma_{c,0}}{\gamma_c}}, \text{ equations (2.22) and (2.20).}$$

3. EMPIRICAL JUSTIFICATION OF THE MODEL

The proposed model is empirically justified with respect to both its hypotheses and its results.

A. *Justification of the Model with Respect to Its Hypotheses*

3.1. *General Model.* The hypotheses (a_1) to (a_4) with respect to the structure of the capitalistic process are rather weak, except for the assumption of homogeneity of order k, but this last characteristic, as I have already pointed out, appears plausible, at least for k equal one.[20]

Hypothesis (b) simply amounts to assuming that it is possible to define production elasticities with regard to primary inputs, and that these elasticities may be considered to vary little in a large region in the vicinity of the process under consideration at a given time and to be fairly constant through time. Both these points appear justified by the results of all previous research in which no variation of the elasticities was assumed, and by the fact that technical progress in itself does not appear to have any systematic influence upon the production elasticities.

3.2. *Exponential Model.* Finally, the hypothesis (c) of exponential decrease of the production elasticities, on which the exponential model is based, appears as rather natural, everything considered. In fact the difficulty of using roundabout processes may be considered as marginally increasing with time in an exponential way.

B. *Justification of the Model with Respect to Its Consequences*

3.3. *General Model. Consequences of its Hypotheses* (a) *and* (b). Without going into the details of the discussion, I simply state that the consequences of the general model are in agreement with the facts *on four important points*:

(1) The practically undetectable variations of consumed real income $\bar{R}_c$ with the capital-output ratio $\gamma = C/R$ for various countries at a given time and for a given country through time.[21] This, a consequence of (2.12) and (2.10*), is confirmed by the available data. (See Allais [1960A, Sec. 48].)

(2) The low variability of the capital-output ratio γ at a given time, for various countries. This is an implication of (2.11).[21] (See Allais [1960A, Sec. 41].)

(3) The low variability of the capital-output ratio γ in time for a given country. This is also an implication of (2.11) if we suppose, as is natural, that $\beta(\theta)$ changes little with time.[21] (See Allais [1960A, Sec. 41].)

[20] In any case, Theorem I remains valid for convex production functions (see footnote 4).

[21] This result becomes particularly clear if we consider the exponential model (relations 2.22, 2.21 and 2.19 corresponding to 2.12, 2.11 and 2.9).

(4) The approximate constancy of the labor value of reproducible capital C/R_ω per unit of primary income at a given time for various countries. This, a consequence of relation (2.9), is confirmed by the available statistical data. (See Allais [1960A, Sec. 43].)

3.4. *Exponential Model. Additional consequences derived from the consideration of hypothesis (c) stating the exponential decrease of $\beta(\theta)$*. Additional consequences derived from introducing the hypothesis (*c*) of exponential decrease of the function β also appear to be empirically verified.

(1) First, hypothesis (*c*) leads for the whole economy to an exponential amortization law (relation (2.16)), which appears to correspond well to the results of empirical research on durable goods depreciation whenever there is an actual market. See, *e.g.*, the study by Boiteux [1956] on second hand cars and that by Blanck and Winnick [1957] on buildings. In these cases fitting exponential curves gives average lives of 6 and 50 years, respectively (Allais [1960A, Sec. 38]).

(2) Secondly, whatever the value of Douglas's results, it is interesting to compare them with the results of the model. Douglas [1948] found for manufacturing industry the empirical relation $P = KL^\lambda C^\mu$ with $\lambda + \mu$ approximately 1 and μ approximately 0.22. These are the average values for the U.S. and Australia for the period 1899–1929.[22] His production function is equivalent to

$$\text{(3.1)} \qquad \frac{dP}{P} = \frac{\mu}{1-\mu}\frac{d\gamma'}{\gamma'} \qquad \text{with} \qquad \gamma' = \frac{C}{P}\,.$$

The exponential model gives (relations (2.23), (2.21) and (2.17))

$$\text{(3.2)} \qquad \frac{d\bar{R}}{\bar{R}} = \frac{i\gamma}{1-i\gamma}\frac{d\gamma}{\gamma}\,.$$

So, the product $i\gamma$ corresponds to the coefficient μ of Douglas's model.

For the period 1899–1929, we can take $\gamma = 3.5$ for i between 4% and 6%, according as we do or do not take into account the rate of growth σ of wages (Table I). Thus we have

$$\text{(3.3)} \qquad 0.14 < i\gamma < 0.21\,.$$

The difference between the values found for μ and $i\gamma$ may be explained by the fact that Douglas's research concerns *industry only*, and *gross* production, whereas our model considers *net* production of the economy *as a whole*. (See Allais [1960A, Sec. 44].)

(3) (4) Two other instances of empirical confirmation may be found (Allais [1960A, Sections 45, 47]) in the good agreement that one finds between

[22] Douglas's results have been very much discussed but, in fact, it seems that they correspond to a very real relationship.

the calculated average amortization time Θ and the labour force composition, and in the agreement between the primary income amortization law derived from the model and the percentage of primary income used in the year's national income.

(5) Finally, my results lead to the conclusion that investment is approximately proportional to income, whatever its level. This conclusion appears to be in good agreement with the results found by Houthakker [1960] that, on the whole, saving is approximately proportional to income. Within the limits of the model, these results appear as consequences of the low variability of the capital-output ratio γ with time and among countries.

4. APPLICATIONS OF THE MODEL

The results I have just stated allow some rather suggestive applications, which are summarized in this section.

4.1. *Possibility of increasing real national income by increasing capital intensity.* On the basis of relation (2.22) and estimating $k = 1$, the relative gain of real consumed income likely to be obtained by realizing the capitalistic optimum is

$$g_c = \frac{\bar{R}_M - \bar{\bar{R}}}{\bar{R}} = \frac{\Theta}{\Theta_0} e^{\frac{\Theta_0}{\Theta} - 1} - 1 \tag{4.1}$$

where, by relation (2.20*), $\Theta = \gamma/(1 - \varrho\gamma)$. As an average for the years 1950, 1955, and 1956, we can take for the United States (Table I) $\gamma = 3.35$, $\varrho = 0.017$, and $\Theta_0 = 3.64$. Therefore, $\Theta = 3.55$ and $g < 0.1\%$. Thus, the gain likely to be realized from increased capital intensity is very low, from which we can conclude that the United States is in the neighbourhood of the capitalistic optimum (Allais [1960A, Sec. 55]).

4.2. *The optimum rate of net saving.* For countries that are in the neighbourhood of the capitalistic optimum, the optimum rate of net saving that one could try to realize is, as a consequence of relation (2.11),

$$\left[\frac{1}{R}\,\frac{dC}{dt}\right]_{i=\varrho} = \varrho\gamma_0 = \frac{\varrho\Theta_0}{1 + \varrho\Theta_0}. \tag{4.2}$$

For the U.S.A., we have $\Theta_0 \approx 4$ and $\varrho \approx 1.7\%$. Thus the optimum rate of net saving is about 6% for the American economy.

4.3. *Influence of the increase of primary income on consumed real income per inhabitant.* On the basis of the general relation (2.14*) the decrease of real income per capita, due to the expansion of primary income, ranges about

$$p = 1 - e^{-k\Theta_0\varrho}\,. \tag{4.3}$$

For the United States during the 1950's we may take $k = 1$, $\Theta_0 = 4$, and $\varrho = 0.017$. Therefore, $p \approx 7\%$, a percentage which is naturally very near the one indicated in the previous paragraph. This loss corresponds to the necessity of assigning a part of national income to maintain the capital-output ratio at a constant value.

4.4. *Explanation of the differences of productivity existing among the different countries.* The present theory allows one to examine in what way differences in capitalistic intensity can explain the average difference in productivity recorded between two countries at a given time.

Thus, in 1955, the average productivity of the American economy was about 2.4 times higher than that of the French economy. One might think at first sight that the explanation of this difference must be sought in the difference in the real volume of equipment. In fact, this volume per worker was approximately 2.4 times higher in the United States than in France, in 1955. Everything indicates, however, that the capital-output ratio γ was at least as high in France as in the United States (in the U.S. it was 3.3; in France, greater). Therefore, the observed difference in productivity cannot be explained by a difference in capital intensity, since the ratios of capital to income have about the same value in the two countries.

This difference must be explained by other factors, such as the differences of natural resources per capita and, over all, the general management of the economy. (See Allais [1960A, Sec. 54] and [1960B, pp. 28–32 and 295–297].) The influence of these other factors produces the result that, for *the same value* of the capital-output ratio, equipment per worker was about 2.4 times higher in the United States than in France, a number *exactly equal* to the ratio of average productivities. Thus we see that the generally admitted explanation amounts to considering *as a cause* a difference in real capital per worker that is in fact *no more than an effect.*

4.5. *The development of underdeveloped countries.* A final suggestive application of the present theory may be made to the case of the so-called underdeveloped countries.

If we accept the hypothesis that the preceding theory is valid, at least as a first approximation, Table III shows in per cent the gain g in real income that could be obtained by increasing the capital-output ratio from the recorded value Θ to the optimum value Θ_0 (relation 4.1), which we can

TABLE III

Θ	4	3.6	3.4	3.2	3.0	2.5	2.0
$i-\varrho$	0	2.8	4.4	6.3	8.3	15.0	25.0
g	0	0.6	1.4	2.7	4.7	13.9	35.9

suppose to be equal to 4. We probably have $\Theta > 2$ for underdeveloped countries.

Then we see that, starting for instance from a situation with a capital-output ratio as low as 2, which corresponds to a difference $i - \varrho$ of about 25%, the gain in real income likely to be obtained by attaining the capitalistic optimum is only about 36%. The possible gain is much lower than is usually thought. Consequently, the explanation of enormous differences in productivity recorded between the Occident and the underdeveloped countries is likely to be found much less in the possible smallness of the capital-output ratio than in differences in natural resources available per inhabitant, differences in the level of technical education, and differences in the management of the economy in general. From this we can conclude that it would not be sufficient to use American type equipment in the so-called underdeveloped countries, if one did not at the same time improve the other factors of production, the productivity of which is so much smaller in those countries than in the Occident.

5. STATISTICAL DATA

In this section we comment further on Tables I and II presented earlier and on further statistical materials.

5.1. *Values of the capital-output ratio in the United States from* 1880 *to* 1956. With regard to Table I, it may be noted that the four values for γ, 3.56, 3.40, 3.56, and 3.35, for the four elementary periods considered are remarkably close to one another and show practically no systematic variation.

The four corresponding period averages for the mean value of Θ_0, 4.34, 4.11, 4.40, and 3.64, are also relatively close and fluctuate in a nonsystematic way around their general average, 4.12. The slightly lower last value, 3.64, may be imputed to the abnormally low values of the rate of interest during the period 1950–56. In fact, the rate on bonds, i_n, went from 2.86% in 1950 to 4.69% in June, 1959, tending thus to come back to its average level of previous periods.

5.2. *Capital-output ratio values in* 1913 *in the United States, France and Great Britain.* With regard to Table II, the concordance of values for Θ_0 for the United States, France and Great Britain in 1913 is as good as might be hoped if we take account of the relative imprecision of the basic statistical data. The average value for Θ_0 agrees well with the four values found for the U.S.: 4.34, 4.23, 4.53, and 3.90 for the periods 1880–1900, 1906–1913, 1923–1937, and 1950–1956, as given in Table I.

5.3. *Capital-output ratios for different countries at different times.* Using data of Colin Clark [1957, pp. 88 ff. and 572 ff.] we find (see Allais [1960A, pp. 54–57]) that, for the lognormal distribution of the coefficient γ' for 58 values for 21 countries considered from 1805 to 1953, the median is 3.54 and the average relative deviation is 22%. The median of 10 values for the U.S. is 3.46.

As far as I can judge, the recorded dispersion of the observed values of γ' may be explained *mainly* by the lack of precision of the data and the differences in calculating methods that are used. In favour of this conclusion, we notice that economically the situation in the United States corresponds to an extreme case. That the corresponding value of γ' is close to the median value of the coefficients γ' of various countries leads one to conclude that essentially the variation of the γ' has a purely statistical origin.

5.4. *Comparison of the estimates of* γ'. We conclude this section with the following comparison of the estimates of γ':

U.S.A. (12 Goldsmith's corrected estimates)	median:[23]	3.46
Year 1913 (3 Allais' estimates)	average:	3.72
World (58 Colin-Clark's estimates)	median:[24]	3.54

The recorded deviations are negligible, if we take account of the lack of precision of the data.

Although Colin-Clark's values are purely illustrative, because of the great differences in the worth of the statistical materials and methods employed, one cannot but be struck by the weak variations of the capital-output ratio, which are shown in § 6.3, for quite different countries and periods with very different standards of living, as well as by the remarkable agreement of the median of the values found with those of both previous groups of estimates for the United States from 1880 to 1956 and for the United States, France and Great Britain for 1913.

In fact, the three values of γ': 3.46, 3.72 and 3.54 are exactly of the same order of magnitude.

CONCLUSIONS

In this paper I have proposed a general theory of the capitalistic optimum, with a model of very general scope which illustrates it, as well as a special case of this general model corresponding to an exponential decrease of production elasticity coefficients.

This general model, and in particular, its exponential variety, appear quite remarkably confirmed by all presently available empirical data, with respect to both the hypotheses and the results.

[23] Empirical value.

[24] Lognormal adjustment.

In fact, *everything takes place as* if the proposed model could be considered as correctly representing concrete reality, and explaining its essential features, very simply.

Whether the research of tomorrow confirms or contradicts this provisional conclusion, I think that in the present state of our knowledge, the proposed theory and the model certainly have the advantage of being simple, suggestive, and forcing one to think about a number of interesting circumstances. After all, this is perhaps the real service which can be rendered by any good theory at a given time.

Ecole Nationale Supérieure des Mines de Paris and Institut de Statistique de l'Université de Paris

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We here present a partially annotated bibliography restricted to works having a direct relation to concepts used in the present paper.

A. *Characteristic Function*

JEVONS, S., 1871: *The Theory of Political Economy.*

In this book, in Chapter VII, the concept of the characteristic function has been used *for the first time* in the case where $\varphi(\theta, i) = \varphi_0$ for $0 < \theta < \lambda$ and $\varphi(\theta, i) = 0$ for $\lambda < \theta$.

BÖHM-BAWERK, EUGEN VON, 1884: *Geschichte und Kritik der kapitalzins Theorien.*

BÖHM-BAWERK, EUGEN VON, 1888: *Positive Theorie des Kapitales.*

In this book, the concept of the characteristic function, among others, has been studied thoroughly (see especially Exkurs I to VI).

BOUSQUET, G. H., 1936: *Institutes de science économique,* Tome III, Rivière, Chap. VI.

This book contains a very noteworthy analysis, for its time, of Jevons' concept of the characteristic function, and the point of view of Böhm-Bawerk is studied thoroughly.

STACKELBERG, VON, 1941: "Capital und Zins in der stationären Verkehrswirtschaft," *Zeitschrift für Nationalökonomie,* vol. X, Feb., 1941.

Relation (1.38) (obtained from the approximate relation $R = R_\omega e^{i\Theta}$) is applied to a particular case (p. 61).

ALLAIS, MAURICE, 1947: *Economie et intérêt.*

To my knowledge, this book gives, for the first time, a general mathematical theory of the characteristic function concept, pp. 118 to 142 and pp. 186 to 206.

B. *Theory of the Optimum Allocation of Resources over Time*

1. *General theory of optimum management*

ALLAIS, MAURICE, 1943: *Traité d'economie pure,* Imprimerie Nationale, 1952 (2nd edition of the first part: "L'Economie pure" of the book *A la recherche d'une discipline economique,* Paris 1943), 5 vol. in-4°, 1,000 pages.

This book gives in chapter VII (pp. 604–682) the first systematic and rigorous statement (to my knowledge) of the theory of the optimum allocation of resources *that takes time into account,* specifies the extremal conditions, *considers the distinction between divisible and indivisible activities, specifies the connection between the conditions* *of optimum allocation of resources and the stability conditions*, and shows clearly the arbitrary nature of the income distribution.

ALLAIS, MAURICE, 1947: *Economie et intérêt*, Imprimerie Nationale, Paris, 1947, 800 pages in two in-8° volumes (in deposit at the Librairie de Medicis).

This book gives in chapter VI, pp. 153–178, the generalization of the theory of optimum allocation of resources *when we consider the successive personalities of the same individual through time, as well as the various generations.*

MALINVAUD, EDMOND, 1953: "Capital Accumulation and Efficient Allocation of Resources," *Econometrica*, April, 1953, pp. 233–268.

This article generalizes the application of some instruments of analysis and gives a summary statement of a certain number of previous results.

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MALINVAUD, EDMOND, 1961: "The Analogy Between Atemporal and Intertemporal Theories of Resource Allocation," *The Review of Economic Studies*, Vol. XXVIII, no. 3.

This article studies some relations between the rate of interest i and the expansion rate ϱ.

ALLAIS, MAURICE, 1961: "L'influence des besoins sur la production des biens de consommation," Communication to the "Colloque de Grenoble," September, 1961, 107 pages.

This paper studies, in Appendix II, the problem of the convexity of production functions.

2. *Theory of the capitalistic optimum for stationary processes*

WICKSELL, KNUT, 1901: *Vorlesungen über Nationalökonomie*; English translation, 1934, *Lectures on Political Economy*, Volume I, "General Theory," Routledge, London, 1946.

The text is rather obscure, but the indication of a capitalistic optimum for $i = 0$ is explicitly given, in a single line (p. 209), in fact, but *for the first time* to my knowledge in the economic literature.

AKERMAN, G., 1923: *Real Kapital und Kapitalzins*, Stockholm.

In spite of a certain obscurity, this work is very penetrating. An interesting commentary is given in the English translation of the previous work, pp. 258–299.

RAMSEY, F. P., 1928: "A Mathematical Theory of Savings," *Economic Journal*, Vol. 38, December, 1928, pp. 543–559.

This well-known paper studies optimum saving for a given generation, but not the capitalistic optimum over time as the present paper does.

KEYNES, J. M., 1936: *The General Theory of Employment, Interest and Money.*

The capitalistic optimum for $i = 0$ is mentioned in a few lines in Chapter XVI, at the beginning of the last paragraph of Section II.

MEADE, J. E., 1937: *An Introduction to Economic Analysis and Policy*, Oxford University Press.

This work gives in Chap. III *the first systematic analysis of the theory of capitalistic* *optimum for* $i = 0$ (see in particular the sixth paragraph); *however, this analysis is presented in a purely literary form.*

KNIGHT, FRANK H., 1946: "Capital and Interest," *Encyclopedia Brittanica*, Chicago, University of Chicago Press, 1946, pp. 799–800, and *Readings in the Theory of Income Distribution*, London, Allen and Unwin, 1950, pp. 384–417.

It is symptomatic that this paper, the purpose of which is to give a general view of the question, makes a very short allusion to the question of a capitalistic optimum, which nevertheless is so important (*Readings*, p. 402).

ALLAIS, MAURICE, 1947A: *Economie et intérêt* (see above).

In Chapter VII, pp. 179 to 228, this work gives the first rigorous statement, to my knowledge, of the theory of a capitalistic optimum as well as the formulation by which one can estimate the order of magnitude of the influence of capital intensity on real income. Among other things, this work establishes, for the case $k = 1$, formulae (1.33) to (1.38).

ALLAIS, MAURICE, 1947B: "Rendement social et productivité sociale," Paper presented at the Washington International Econometric Congress in 1947; *Proceedings of the International Statistical Conferences*, 1947, Vol. V, pp. 129 ff.

This paper summarizes some results of the book, *Economie et intérêt.*

ALLAIS, MAURICE, 1949: *Intérêt et productivité sociale*, Thesis of "Ingénieur-Docteur de l'Université de Paris," Faculté des Sciences, June, 1949.

MALINVAUD, EDMOND, 1953: "Capital Accumulation and Efficient Allocation of Resources" (see above).

This article gives a new demonstration of the capitalistic optimum for $i = 0$, with reference to previous works.

KOOPMANS, T. C., 1957: *The State of Economic Science*, McGraw-Hill.

This book gives some indications, although very succint, about the theory of a capitalistic optimum with only one reference to previous works: Malinvaud's article (1953).

ALLAIS, MAURICE, 1960B: *L'Europe unie, route de la prospérité*, Calmann-Lévy, Paris, 1960, 368 pp.

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ALLAIS, MAURICE, 1961B: *Le Tiers-Monde au carrefour*, Les Cahiers Africains, Créations de Presse, vol. 7 and 8, Bruxelles, 1962.

In one of its chapters, this book applies the theory of a capitalistic optimum to the case of underdeveloped countries (pp. 52–53).

3. *Capitalistic optimum theory for dynamic processes*

DESROUSSEAUX, JACQUES, 1959: "Variations sur la croissance économique," 16 pages and an Appendix of 13 typewritten pages.

For the first time, to my knowledge, this study sets forth, in a precise way, the capitalistic optimum condition $i = \varrho$ for dynamic processes in the case of a particular model.

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The equality, $i = \varrho$, is indicated as a general condition for a capitalistic optimum (M. Desrousseaux's priority on that proposition is stated, p. 6).

DESROUSSEAUX, JACQUES, 1961A: "Expansion stable et taux d'intérêt optimal," *Annales des Mines*, November, 1961, pp. 31 and 46.

By generalizing Allais' formulation (1947A and 1947B) to the case of a non-stationary process, and by assuming that the function $\varphi(t, \theta)$ satisfies a "condition of regularity" over time, this study establishes the condition for a capitalistic optimum, $i = \varrho$. Although questionable in some of its parts, this study presents original and very suggestive points of view.

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C. *Allais' Model Illustrating the General Theory of a Capitalistic Optimum*

1. *Stationary processes*

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2. *Generalisation in the case of a dynamic process and of any function* $\beta(\theta)$

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This study applies the method of Desrousseaux's paper (1961A) to the case of a function $\beta(\theta)$ independent of time. It leads to results very similar to ours and which are identical if the rates i and ϱ are constant. The results obtained are in general different, for M. Desrousseaux establishes his results by starting from a condition of regularity whereas ours are derived from Theorem I. This paper of M. Desrousseaux is prior to my demonstration of Theorem I.

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This note generalizes the results of Allais' Tokyo paper (1960A) for rates i and ϱ constant.

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D. *Empirical Research*

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