Stable matchings in an economy with strong and weak agents*

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Abstract

A simple model of matching between two populations is proposed. Agents search for partners from the other population to establish a pair interaction that brings them profit. Within each population the agents differ in their probabilities of exit from the economy. The composition of every pair determines its expected lifetime and profits the agents have from the interaction. The agents' optimal decisions of accepting or rejecting a match are studied in a stationary state of the economy. It is shown that several types of stable matching can occur depending on the matching technology and the composition of types of agents in the economy.

*This version is preliminary and incomplete.

1 Introduction

This paper is inspired by the research that has been done on equilibrium properties of two-sided matching markets. Traditionally, labor market or marriage market are studied. These markets are considered to be two-sided because the matches are created between agents from two distinct disjoint populations. Good examples of such populations are firms and workers, men and women. The match is defined as a long-term relationship of two agents, each coming from a different population, and it is assumed that being in a match is a profitable activity. The economy is studied in a stationary situation when the matching between the two populations is stable, i.e. no matched individual prefers to be single, and no single individual, when having an opportunity to match, would choose the match over the option of staying single.

This paper can be motivated as a description of a matching market in the economy where agents create social contacts. These can be friendship, business partnership, or marriage. It is assumed that people are heterogenous in their abilities to maintain social contacts, i.e. they differ in their probabilities of leaving the match, which, by assumption, means leaving the economy. Agents' probabilities of exit influence lifetime of matches in the economy and profit from the matches. The proposed model allows to study equilibrium properties of the matching market with entry and exit of agents. Optimal individual decisions of accepting or rejecting each particular type of match are analyzed and social optimality of these decisions is assessed.

In the past, two-sided matching models have been used to address questions related to labor markets and marriage markets. The two sided matching models have been widely used in both macro and microeconomic literature. Mortensen and Pissarides [7] developed a model of two-sided matching between vacant jobs and unemployed workers. The model was able to explain reasonably well the job creation and job destruction observed in U.S. The Mortensen-Pissarides aggregate matching function was widely used in macroeconomic models of job search.

A model of matching between employers and workers by Kiyotaki and Lagos [5] helped to explain empirically observed percentage of job-to-job transitions.

In micro-oriented literature, the discussion on two-sided matchings started by the "marriage" model of Gale and Shapley [4]. They assumed that every man has preferences over women and every woman has preferences over men and they studied properties of the set of stable matchings in the economy. The "marriage" model was then extended in many ways, especially by assuming different degrees of transferability of the utility within pairs (e.g. Burdett and Wright [2]).

An interesting two-sided matching model was proposed by Burdett and Coles [1]. They assumed that the agents are ex-ante heterogenous, each is characterized by a real number which is in fact the utility of the spouse after they agree to marry. In this setting the authors were able to observe an equilibrium sorting of agents into clusters based on the numbers by which they are characterized. Burdett and Coles focused their attention only on the process of match creation, i.e. once a match is created the agents leave the market and are replaced by new agents.

The following model also focuses on the process of match creation. The agents come from two populations and they are assumed to be of two types within populations - strong agents and weak agents. The types differ by their exogenous probabilities of leaving the market. Since, by assumption, the matched agents do not have an opportunity to meet other agents, the matches split only due to exogenous reasons, i.e. when one of the partners exits the economy. The strength of the agents has therefore a direct impact on the expected lifetime of a match.

Matching enables interaction of agents, which is modelled like a production process. Proceeds from the production are then split between the members of the pair. Single agents can not interact but they have prospects of being matched in the future. Optimal behavior of the agents imply that a match is created only when both partners find it profitable, taking into account the outside option of staying single. Once created, a match is stable. The matched agents do not have any incentives to walk away because their outside options do not change over time. Agents' optimal decisions of creating or rejecting a match are studied in an environment where agents differ only in their probabilities of exit from the economy but not in the productivity.

Possible extension of the proposed model are to allow agents search while matched or to allow for interaction of more than two agents. These extensions may serve as models of search on the labor market.

2 The Model

Time in the economy is discrete and the horizon is infinite. The economy is populated by two disjoint populations of agents. The populations are completely identical. Each population is of mass $\frac{1}{2}$. Ex ante there are two (observable) types of agents in each population - strong ones and weak ones. The mass of the strong ones per one population is $\frac{A}{2}$ and the mass of the weak ones is $\frac{1-A}{2}$. The agents are characterized by their strength, which is measured by their probability of exit from the economy. Let w denotes the probability of exit of a weak agent in a given period and s the probability of exit of a strong agent in a given period ($w > s$).

The interaction between agents of the two populations, which is modelled like a production process, is happening in pairs. Assume that the production is such that a pair of agents produces $2 \cdot \pi$ units of goods which they split. There are two cases to be considered. In the first one the agents bargain over splitting the proceeds from production. I will refer to this case as the bargaining case. The other case is the one when the agents split the proceeds equally. That is the no-bargaining case.

Agents in the economy can be either single or matched. Those that are matched produce every period until one member of the pair exits. The other member is left single and the exiting one is replaced by a single agent of the same type, i.e. the agent from the same population and of the same strength as the exited one. Note that since no new information is revealed over time and the agents can not search for a new partner while matched, the agents do not have any incentives to walk away from the match once it was formed, i.e. the exits in the economy are only exogenous.

Single agents enter a market for singles. On this market a fraction m of the singles is randomly proposed matching into pairs. The agents then individually choose whether to accept such a proposed match or stay single for another period. The conditions under which the agents accept the match in both bargaining and no-bargaining cases are discussed. The analysis is performed with respect to four parameters: the fraction of strong agents in the population A, the probability of being matched when single m, the probability of exit from the economy of the weak agents w, and the probability of exit from the economy of the strong agents s.

Clearly, there are six types of agents in each population. Three types of strong agents: a strong one matched with a strong from the other population, denoted ss type; a strong one matched with a weak one, sw type; and a non-matched strong, so type; and similarly three types of weak agents: ws type, ww type, and wo type.

2.1 The Bargaining Case

In this subsection the situation in which the agents have the possibility to bargain over the proceeds from production is analyzed. When two agents of a different strength meet they will enter a bargaining procedure. The bargaining takes into account that the outside option of agents is to stay single for another period. When the agents in a match are of a different strength their values of being single differ and therefore the split of the proceeds from production will be uneven.

As was already stated, the one-period production of a pair is independent of the composition of a pair. The pairs differ only in their expected lifetime, which depends on the composition of the particular pair. Therefore the pairs differ in their expected profits.

The agents do not have any decisive power over the production, the only decision the agents face is whether to match with a proposed partner when the matching situation occurs. Intuitively it is in the interest of both sides to match because only the pair interaction brings profits to the agents. But due to bargaining it can happen that not every proposed match is accepted. The basic trade-off of the model is between the expected profit from a match and the expected lifetime of a match. Based on this trade-off the agents may be willing to reject a certain proposed match. As an example, consider the case when the probability of matching in each period is relatively high (m is high), the fraction of weak agents in the economy is high (A is small), and the difference between the strength of weak and strong agents is big ($w - s$ is big). Then when two strong agents meet they may consider to reject the proposed match because they know that in the next period they have a high chance to be matched with weak agents and because of the big difference between the strengths they will be able to extract a lot of profits from the weak agents.

The bargaining is in fact a take-it-or-leave-it offer. When two single agents meet one of them is randomly chosen to suggest the split of the proceeds from future production of the pair (each of the agents has probability $\frac{1}{2}$ to be chosen). This agent will offer to his counterpart the smallest share possible so that the counterpart still takes the offer, i.e. the profit the counterpart would have today when taking an outside option of staying single. The value of the outside option is the discounted value of being single (of the corresponding type) in the next period (it is discounted by the time factor β but also by the probability that the agent survives till the next period). But this value sums all the future profits of the agent of a particular type. The proposing agent will offer only the part of this value that corresponds to the present period.

This means that, for example, in the case of the match of a strong and a weak agent when strong agent is the proposer, the outside value of his weak counterpart is $\beta(1-w)v_{wo}$, where v_{wo} denotes the value of being wo type at the beginning of every period. The value is a sum of all expected future profits of a weak single agent that come from the possible matches in the future. The exact formula for the value will be stated later on.

The outside option differs from the value v_{wo} because the agent was in the present period already proposed a match and if he rejects it his profit in the present period is 0 and the value comes only from the future prospects given the agent will survive till the next period. The present's part of the outside value is $(1-\beta(1-w))\beta(1-w){v_{wo}}$[1]. The strong agent therefore has to offer $(1-\beta(1-w))\beta(1-w)v_{wo}$ to the weak one and he will take $2\pi-(1-\beta(1-w))\beta(1-w)v_{wo}$. This happens with probability 1/2. With the same probability he will get $(1-\beta(1-s))\beta(1-s)v_{so}$ when the weak one is the proposer, and the weak one will take $2\pi-(1-\beta(1-s))\beta(1-s)v_{so}$.

Note that when agents of the same type are matched they both have the same bargaining power and therefore they must split the proceeds of the

[1] $(1-\beta(1-w))\beta(1-w)v_{wo}(1+\beta(1-w)+(\beta(1-w))^2+(\beta(1-w))^3+...)=\beta(1-w)v_{wo}$

production equally, i.e. both agents get exactly π. Also note that the split of profits as described above is in fact the Nash bargaining result.

Because no searching while matched is allowed, the agents have no incentives to walk away from a match once they have accepted the match. Moreover, every period a particular type of agent faces the same prospects. Consequently, we can express the value of being certain type recursively. The values for the six types [2], under the assumption that each agent would accept the proposed match, are:

$$v_{ss} = \pi + \beta\Big((1-s)^2 \cdot v_{ss} + (1-s)s \cdot v_{so}\Big)$$

$$v_{sw} = 1/2\Big(2\pi - (1-\beta(1-w))\beta(1-w) \cdot v_{wo} + (1-\beta(1-s))\beta(1-s) \cdot v_{so}\Big) + \beta\Big((1-s)(1-w) \cdot v_{sw} + (1-s)w \cdot v_{so}\Big)$$

$$v_{so} = m \cdot \Big(S \cdot v_{ss} + W \cdot v_{sw}\Big) + (1-m)\beta(1-s) \cdot v_{so}$$

$$v_{ww} = \pi + \beta\Big((1-w)^2 \cdot v_{ww} + (1-w)w \cdot v_{wo}\Big)$$

$$v_{ws} = 1/2\Big(2\pi - (1-\beta(1-s))\beta(1-s) \cdot v_{so} + (1-\beta(1-w))\beta(1-w) \cdot v_{wo}\Big) + \beta\Big((1-w)(1-s) \cdot v_{ws} + (1-w)s \cdot v_{wo}\Big)$$

$$v_{wo} = m \cdot \Big(S \cdot v_{ws} + W \cdot v_{ww}\Big) + (1-m)\beta(1-w) \cdot v_{wo}$$

where we assume that the value of exit is 0, β is a factor by which agents discount future.

Since the Law of Large Numbers holds, in the matching situation the agents will face a weak or a strong counterpart with probabilities that are proportional to the fractions of weak and strong agents that are single. The probabilities, and also the fractions of weak and strong agents in the pool of single agents, are denoted W and S respectively.

[2] The values are irrespective of which population the agent of a particular type comes from.

The probability S is equal to

$$S = d_{so}/(d_{so} + d_{wo})$$

and the probability W is equal to

$$W = d_{wo}/(d_{so} + d_{wo}) = 1 - S$$

where $d_{..}$ are distribution fractions of agents of indicated types[3].

The system of value functions can be rewritten in a matrix form as

$$\mathbf{V} \cdot \mathbf{v} = \boldsymbol{\pi}$$

where $\mathbf{v'} = (v_{ss}, v_{sw}, v_{so}, v_{ws}, v_{ww}, v_{wo})$, $\boldsymbol{\pi'} = (-\pi, -\pi, 0, -\pi, -\pi, 0)$ and $\mathbf{V}$ is a matrix implied by the system of equations. The matrix equation can be analytically solved to get the value functions dependent only on parameters of the model.

$$\mathbf{v} = \mathbf{V}^{-1} \cdot \boldsymbol{\pi}$$

The distribution of agents across types $\mathbf{distr} = (d_{ss}, d_{sw}, d_{so}, d_{ws}, d_{ww}, d_{wo})$ evolves in time according to the vector equation

$$\mathbf{distr_{t+1}} = \mathbf{distr_t} \cdot \mathbf{Q}.$$

$\mathbf{Q}$ is a transition matrix that describes movement of agents across the states.

We are looking for a stationary distribution $\mathbf{distr}^*$, i.e. distribution that is stable in time

$$\mathbf{distr}^* = \mathbf{distr}^* \cdot \mathbf{Q}$$

Since the agents are ex-ante of two strengths we will look for two stationary distributions, one for each type. Note that the fraction of ws type in the

[3]Note that since the two populations in the economy are completely identical we can do all the distribution computations for the whole economy at once and then multiply all the fractions of distribution by $\frac{1}{2}$ to obtain the results for each of the two populations.

stationary distribution of weak agents must be the same as the fraction of sw type in the stationary distribution of strong agents.

Under the assumptions that the Law of Large Numbers holds and every agent accepts the proposed matching the transition matrices for strong and weak agents are QS and QW. The interpretation is that an element q_{ij} is the probability that the next period the agent will be of type j given that today he is of type i.

$$QS = \begin{pmatrix} (1-s)^2 & 0 & (1-s)s+s \\ 0 & (1-s)(1-w) & (1-s)w+s \\ mS(1-s)^2 & mW(1-s)(1-w) & \substack{(1-m)+mS((1-s)s+s) \\ +mW((1-s)w+s)} \end{pmatrix}$$

$$QW = \begin{pmatrix} (1-w)^2 & 0 & (1-w)w+w \\ 0 & (1-s)(1-w) & (1-w)s+w \\ mW(1-w)^2 & mS(1-w)(1-s) & \substack{(1-m)+mS((1-w)s+w) \\ +mW((1-w)w+w)} \end{pmatrix}$$

The types are ordered ss, sw, so in QS matrix and ww, ws, wo in QW matrix.

When computing the stationary distributions of weak and strong agents we have to take into account that $d_{ss} + d_{sw} + d_{so} = A$, and $d_{ww} + d_{ws} + d_{wo} = 1 - A$ must hold.

It is important to note that S and W are functions of the stationary distribution, therefore the system of equations describing the stationary distribution is not linear.

2.2 The Matching Strategy

In the previous section the system of values of six possible types of agents was analytically described. The system of equations for the values can be analytically solved and the results that are only the functions of the parameters of the model can be obtained. The equilibrium fractions of single agents enter the computation of the values through the terms W and S and they complicate the analytical expressions for the values. The analysis of the dependence of agents' values on the parameters of the model is therefore done through a simulation rather than through an analytical derivation that would be extremely tedious considering the complexity of expressions for the values of agents.

The matching strategy of the agents depends on the values the agents have when being a certain type. Because the agents do not change their strength over time, an agent of a particular strength can be only of three types. The agent can be matched with a strong counterpart, with a weak counterpart or the agent can be single. The values of these three types can be ordered in 6 possible ways. And since the agents are of two strengths we have 36 possible ordering combinations of the values. In all those orderings where the value of being single is higher than one of the values of being matched, the match to that particular type will be rejected. Note that it is enough that one counterpart rejects to create the match and then both counterparts stay single for another period.

As an example, let the equilibrium ordering for a certain range of parameters be $v_{sw} > v_{ss} > v_{so}$ and at the same time $v_{ww} > v_{wo} > v_{ws}$. Clearly, if this ordering holds the proposed matching is unstable. The weak agent who is proposed a match with a strong one will reject to match because the value of staying single is higher. So there will be no ws-type agents in the equilibrium. Consequently, there will be also no sw-type agents in the equilibrium. Knowing that, the whole computation of the stationary distribution as well as the value functions must be redone under the assumption that the types ws and sw do not exist.

In practice, the transition matrices will change in their third lines. If a single agent meets an agent of the strength other than his, he will reject the match and stay single. Consequently, there will be no in-flow of agents of ws and sw types, so the equilibrium fractions of these types will be zero.

Of course, since the equilibrium distribution will be different than in the case of accepting all the proposed matches, the values of types will change. For the computation of the values the system of equations from the previous section can be used, but the fractions S and W have to be changed based on the computation of the new equilibrium distribution and also the assumption that the values v_{ws} and v_{sw} are equal to zero should be used.

The computations of equilibrium distribution and values can be done in a similar manner for all the orderings of the values where some of the proposed matches would be rejected.

Some of the orderings of the values of the thirty six possible can be immediately ruled out as impossible based on the structure of the model. These are, for example, the orderings where the value of being single of a particular strength would be higher than both values of being matched. Clearly this can not be the case because the value of being single is a linear combination of the values of being matched and itself and the multipliers of the values sum to a number smaller than 1. This consideration rules out twenty orderings of the values.

From the remaining sixteen orderings some are such that every proposed match is accepted and some are such that certain types of matches are rejected.

All the matches are accepted for 4 possible orderings, in the remaining twelve cases some of the agents would prefer to stay single rather than match with a proposed partner. For these cases the computation of equilibrium distribution and equilibrium values should be done once again because the rejection of a certain type of match will have an impact on the distribution and through it also on the values of the types.

As already mentioned, it can never happen that both matching with a weak agent and matching with a strong agents would be rejected at the same time (it would mean that the value of staying single is the highest).

It is also clear that when the pairs of agents of unequal strength are rejected by one of the counterparts, then there are no mixed pairs in equilibrium and after recomputing the values all the proposed matches must be accepted. It follows from the fact that for both strong and weak agents they can be either matched with their own type or they can be single. In such situation the value of being single must be smaller than the value of being matched because the basic trade-off of the model between the expected lifetime of a pair and the expected profit from a pair has disappeared.

Other types of results that can occur are, for example, the rejection of strong agents to match with agents of the same strength. If all the other proposed matches are accepted, the equilibrium will consist of five types of agents. Another example can be a combination of the previously mentioned cases, i.e. agents of one strength will reject to match with the agents of the other strength and at the same time these agents will reject to match with their own type. Such rejections then lead to a situation when the agents of the second strength do not match at all, which is clearly not socially optimal since matching with own type would be profitable once the matching with the opposite type is unfeasible.

The results of the model will show which of the cases described above occur as equilibria of the model and under what conditions a particular type of equilibrium occurs.

2.3 The No-Bargaining Case

This section describes a situation in which agents do not have a possibility to bargain over proceeds from their production. In this case every couple splits the profit equally. This case, though it is not very interesting, is discussed to complete the analysis of the model. Because one-period profit from a match is independent of the composition of the match, the matches differ only in their expected lifetime and therefore there exists a unique ordering of values of the agents in equilibrium that does not depend on the values of parameters of the model.

In a similar manner as in the bargaining case, the values of being a certain type can be written in a recursive way:

$$
\begin{aligned}
v_{ss} &= \pi + \beta\Big((1-s)^2 \cdot v_{ss} + (1-s)s \cdot v_{so}\Big) \\
v_{sw} &= \pi + \beta\Big((1-s)(1-w) \cdot v_{sw} + (1-s)w \cdot v_{so}\Big) \\
v_{so} &= m \cdot \Big(S \cdot v_{ss} + W \cdot v_{sw}\Big) + (1-m)\beta(1-s) \cdot v_{so} \\
v_{ww} &= \pi + \beta\Big((1-w)^2 \cdot v_{ww} + (1-w)w \cdot v_{wo}\Big) \\
v_{ws} &= \pi + \beta\Big((1-w)(1-s) \cdot v_{ws} + (1-w)s \cdot v_{wo}\Big) \\
v_{wo} &= m \cdot \Big(S \cdot v_{ws} + W \cdot v_{ww}\Big) + (1-m)\beta(1-w) \cdot v_{wo}
\end{aligned}
$$

Also in this case the system of equations can be solved analytically. Moreover, the system can be separated into two systems of three equations. The transition matrices are the same as in the bargaining case when the agents accept all the matches proposed.

Because the agents do not have a possibility to bargain over the profit from production the sw-type agents are not compensated for the fact that their partners die more often then the partners of ss-type agents. The question then is whether it is indeed profitable to match with the weak type or it is better to stay single for another period and wait for the match with a strong agent.

This question can be easily answered even without any computations. If the strong agents accept the match with the weak ones the worst case scenario is that in the given period their profit is π and immediately in the next period their weak partner exits, leaving the strong one single. This scenario is definitely more profitable than rejecting the match because by rejection the strong agent is losing this period's profit and in the next period he will find himself in the same situation as if he accepted the match and his partner exited immediately.

Because in the no-bargaining case the trade-off between the profit from a match and the lifetime of a match disappears it is clear that the most profitable match is the one with a strong counterpart, then follows the match with a weak counterpart and the worst case is to stay single. This ordering of the values of types holds both for strong and the weak agents, i.e. $v_{ss} > v_{sw} > v_{so}$ and at the same time $v_{ws} > v_{ww} > v_{wo}$.

3 Results

The values of all types of agents can be analytically expressed as the functions of parameters of the model. The values are homogenous of degree 1 in π.

Although analytical expressions for value functions and also fractions of distributions can be obtained, the expression are so complicated that it is not clear how to analyze them analytically. That is why some parts of this section rely on numerical simulations.

The following results for the stationary distribution hold for the bargaining case when all the matches are accepted and for the no-bargaining case (the stationary distribution is the same in these cases).

When solving for the distribution, the system of six equations together with two constraints, $d_{ss} + d_{sw} + d_{so} = A$ and $d_{ww} + d_{ws} + d_{wo} = 1 - A$, can be narrowed down to the system of two quadratic equations with two unknowns that has two sets of solutions. It can be shown that only one of these will give positive results for all combination of parameters s, w, m, A. The system of quadratic equations is this one:

$$d_{so}^2(1-m+\frac{m}{2s-s^2})+d_{so}d_{wo}(1-m+\frac{m}{s+w-sw})-Ad_{so}-Ad_{wo}=0$$

$$d_{wo}^2(1-m+\frac{m}{2w-w^2})+d_{so}d_{wo}(1-m+\frac{m}{s+w-sw})-(1-A)d_{so}-(1-A)d_{wo}=0.$$

Then the solution of the system[4] that is plausible, i.e. that gives positive fractions of d_{so} and d_{wo}, is this:

$$d_{so}=\frac{-xy+y^2-2Ay^2+2Axz+(x-y)\sqrt{(1-2A)^2y^2-4(-1+A)Axz}}{2x(-y^2+xz)}$$

$$d_{wo}=-\frac{y^2-2Ay^2-2xz+2Axz+yz+(y-z)\sqrt{(1-2A)^2y^2-4(-1+A)Axz}}{2z(-y^2+xz)}$$

where x, y, z stand for

$$x=1-m+\frac{m}{2s-s^2}$$

$$y=1-m+\frac{m}{s+w-sw}$$

$$z=1-m+\frac{m}{2w-w^2}$$

The other four fractions of the stationary distribution can be expressed, using d_{so} and d_{wo}, like this:

$$d_{ss}=m\cdot\frac{d_{so}^2}{d_{so}+d_{wo}}\cdot\frac{(1-s)^2}{2s-s^2}$$

$$d_{sw}=d_{ws}=m\cdot\frac{d_{so}d_{wo}}{d_{so}+d_{wo}}\cdot\frac{(1-s)(1-w)}{s+w-sw}$$

$$d_{ww}=m\cdot\frac{d_{wo}^2}{d_{so}+d_{wo}}\cdot\frac{(1-w)^2}{2w-w^2}$$

[4]The computation is done for both male and female populations together, so the to get the distribution corresponding to one of the populations all the results should be multiplied by $\frac{1}{2}$.

As pointed out earlier, despite the stationary distributions are the same in the bargaining case when all the matches are accepted and no-bargaining case, the two cases differ when value functions are considered. It is clear that in the no-bargaining case the agents' value is the largest when matched with a strong type, and the lowest when single. This does not have to be true in the bargaining case where the agents can extract profits when matched with an agents of a different strength than theirs.

The bargaining case is very interesting because it suggests a lot of questions to which we do not have even intuitive answers. For example: How much can the strong agents extract from weak agents when matched with them? Is it possible that the value of an sw type can be higher than the value of an ss type and if yes then how is this result dependent on the values of the parameters? For what range of parameters the agents refuse to match with the opposite type? Do agents' matching decisions lead to a socially optimal result?

The following analysis is done based on the results of a simulation. In the simulation the profit π is fixed and is equal to 1. Since the values are homogenous of degree 1 in π this choice of numerical value of π is not important. The discount factor is $\beta = 0.95$, which is a standard value. The simulation is performed for different fixed values of A and m between zero and one, namely for eleven cases: for every 0.1 point in the interval $(0, 1)$ and for the extreme cases 0.001 and 0.999. To simulate a continuous range of values of $w \in [0, 1]$ and $s \in [0, 1]$ the computations are done for every 0.02 point between zero and one. Note that in the model it is assumed that $w > s$ and the simulation reflects that. For the simulation the analytical results for the equilibrium distribution of agents are used and these are then entering a numerical computation of the values for all possible combinations of the parameters of the model A, m, w, and s.

The goal of the simulation is to determine how many orderings of the values of different types of agents the model can produce. Based on the orderings different types of equilibria are described.

The analysis proceeds as follows. For each combination of the parameters of the model the values of agents are ordered. Then the combinations of the parameters are divided into groups based on what type of ordering of the values they imply. The same ordering of the values means the same type of equilibria. Unfortunately, patterns of different types of equilibria are difficult to describe analytically so the whole comparative statics exercise is done based on plots of different type of equilibria. The plots are very illustrative and can provide a clear intuition about the results of the model.

For a fixed value of m, which is a probability of being matched with a partner when single, the evolution of different types of equilibria is studied when the parameter A, which is the fraction of strong agents in the economy, is growing. The types of ordering are then plotted for all combinations of the values of w, which is the probability of exit of the weak agents, and s, which is the probability of exit of the strong agents.

The simulation shows that there are seven possible orderings of the values of agents. For each ordering a different type of plot is used:

1. Vss > Vsw > Vso & Vws > Vww > Vwo
2. Vss > Vsw > Vso & Vww > Vws > Vwo
3. Vsw > Vss > Vso & Vww > Vws > Vwo
4. Vsw > Vso > Vss & Vww > Vws > Vwo
5. Vss > Vsw > Vso & Vww > Vwo > Vws
6. Vsw > Vso > Vss & Vww > Vwo > Vws
7. Vsw > Vss > Vso & Vww > Vwo > Vws

The first three types of orderings are stable, i.e. all the matches are accepted, because the values of being single are the lowest for agents of both types of strength. Each of these orderings is therefore one type of equilibrium characterized by a stationary distribution of agents (which is the same in all three cases) and by values of each type of agents.

The orderings $4.-7.$ are unstable because there is always at least one type of agents that would prefer to stay single rather than to be matched with a proposed partner. In particular, in the case of the 5th and 7th ordering, the weak agents reject to match with the strong agents, which implies that there are no mixed pairs in equilibrium. These two orderings therefore lead to one type of equilibria - the equilibrium with 4 types of agents: single agents and agents matched with a counterpart of the same strength.

The 4th type of ordering is such that the strong agents will reject to match with another strong. In this case a further computation is needed in order to see whether after the rejection of the ss-type of match all the other types of matches are stable. If yes, then the equilibrium with five types of agents would be obtained. This case is discussed later on.

The 6th type of ordering is probably the most interesting one. It suggests that the strong agents reject to match with their own type and at the same time the weak agents do not find it profitable to match with strong agents which leads to a situation when all the strong agents stay single. Clearly, it is socially not optimal. The profit of the agents and the whole economy would be higher if the strong agents were matched with their own type.

The analysis of the evolution of the 7 types of orderings and the equilibria they imply is done based on the plots obtained from the simulation. Cases are discussed for a particular value of the probability of being matched m. Figure 1 is a representative plot of the equilibria in the case of m being extremely small. The plot is basically the same no matter what is the share of strong agents in the population. That is why the plot is presented for the most natural share equal to 0.5. Note that the vertical axis represents the probability w while the horizontal axis represents s, and $w > s$ must hold.

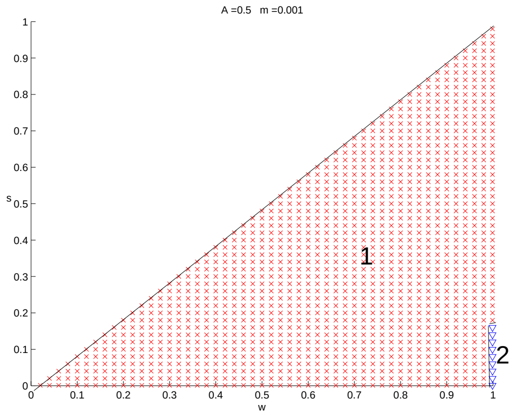


Figure 1: m = 0.001 and A = 0.5

When the probability of matching is extremely low all the matches are accepted and the value of being matched with a strong agent is higher than the value of being matched with a weak agent. The values of staying single are the lowest. This is the ordering 1. For the value of the probability w approaching to 1 and the values of the probability s that are relatively much lower the value of a weak agent when matched with another weak will become larger than the value of being matched with a strong agent, i.e. the strong agents will extract a lot of profits from the fact that the difference $w - s$ is big and this will drive down the value v_{ws}, i.e. the ordering 2 occurs.

Similar pattern can be observed when the probability of being matched increases to 0.1. All the proposed matches are accepted, the region of the ordering 2 gets larger and this region gets larger also with the increasing fraction of strong agents in the economy, as illustrated by Figure 2.

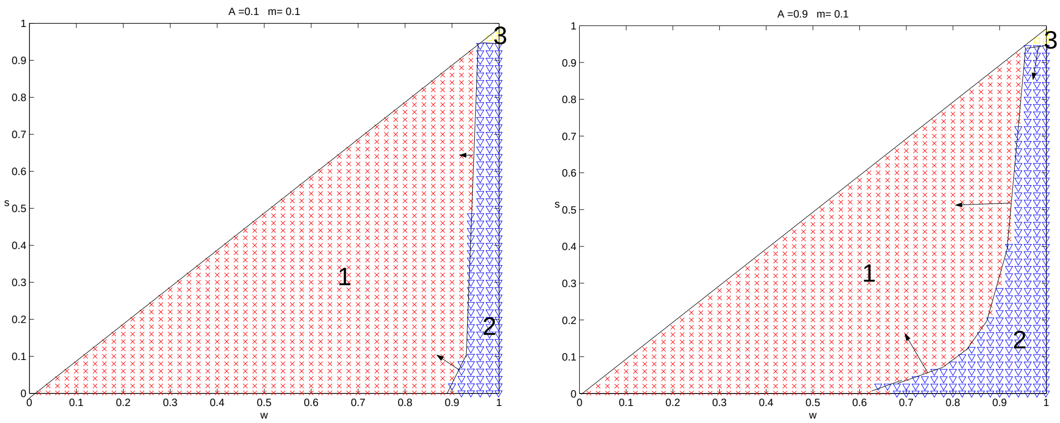


Figure 2: m = 0.1 and A = 0.1, A = 0.9

A new type of ordering, the ordering 3, emerges for the values of both probabilities of exit being close to one. All the proposed matches are accepted but now the value of a strong agent being matched with a weak one exceeds the value of being matched with another strong agent. Naturally, we can expect this equilibrium to arise in a situation where w is close to s because then the lifetime of a match is approximately the same, but the strong agent can still extract profits from a weak one based one their different values of being single.

With the increasing probability of being matched, i.e. with m increasing, the orderings of the values of agents 1, 2, and 3 are still the only possible orderings, i.e. still all the proposed matches are accepted. The ordering 2 becomes prevailing. As it can be seen on Figure 3 the ordering 3, typical for the values of s and w close to each other, now holds not only for the values of probabilities close to 1 but also for the probabilities close to 0. The area of the ordering 1 shrinks as m increases.

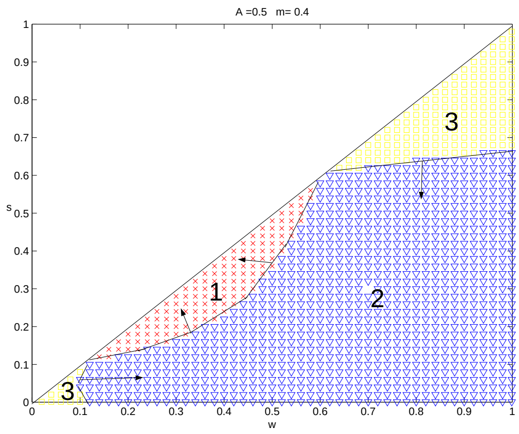


Figure 3: m = 0.4 and A = 0.5

For the probability of being matched equal to 0.5 all the proposed matches are still accepted but the ordering 1 of the values of the types completely disappears. The regions of the ordering 2 and 3 are of the shape as shown on the Figure 4 and they are almost not changing with an increase of the share of strong agents in the economy A.

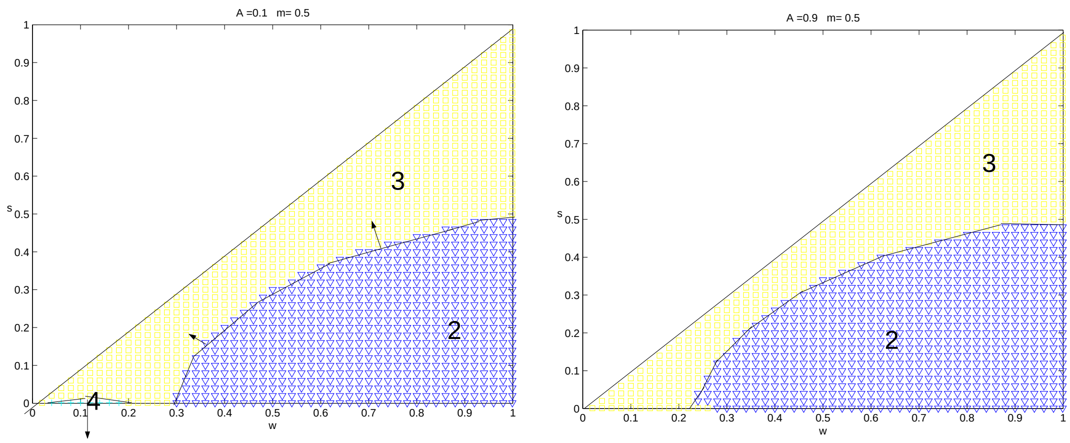
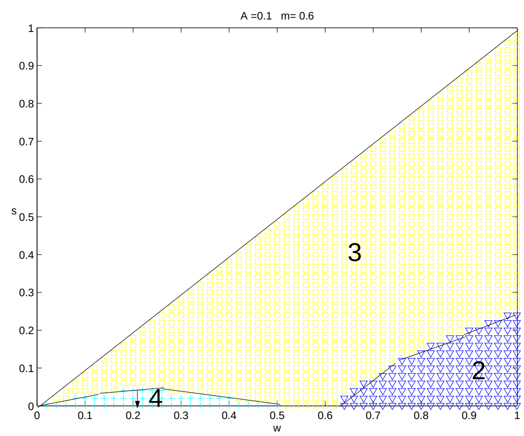


Figure 4: m = 0.5 and A = 0.1, A = 0.9

In general, the first conclusion is that for $m \leq 0.5$, i.e. when the matching possibilities are relatively rare, all the proposed matches are accepted. Once the probability of being matched increases the agents may want to reject some matches with a low expected profit because they can count on the fact that, in expectation, soon they will have a chance to accept a more profitable match.

As Figure 5 shows, the ordering 4, which implies a rejection of ss-type pairs occurs for the first time for the matching probability being higher than fifty percent and the share of the strong agents in the economy being relatively small. The Figure 5 also shows a region of the ordering 5 which occurs only for one particular (very extreme) combination of parameters. When there are almost no weak agents in the economy and their probability of exit is very high in comparison with the probability of exit of the strong agents, then it is indeed possible that the weak agents will choose to reject matching with strong agents who are extracting a lot of profits from the big difference in the probabilities of exit and at the same time the value of the ss-type of agents will be still higher than the value of being the sw-type.



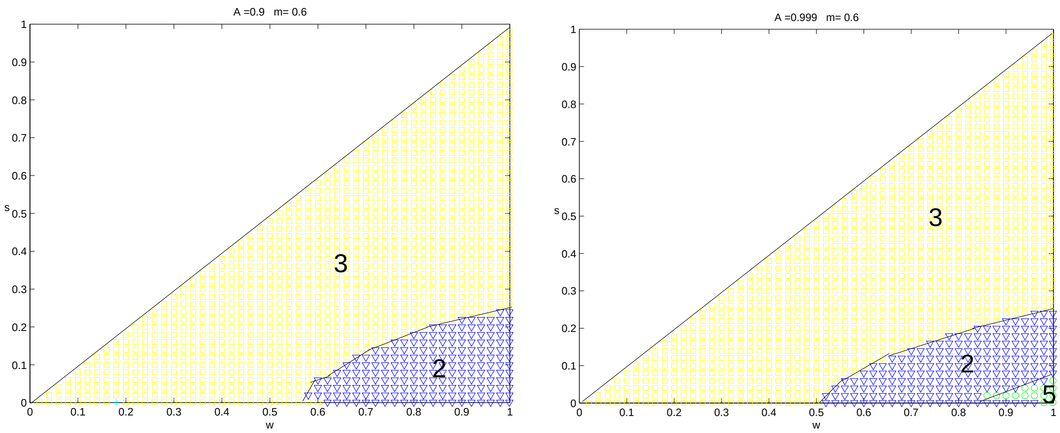
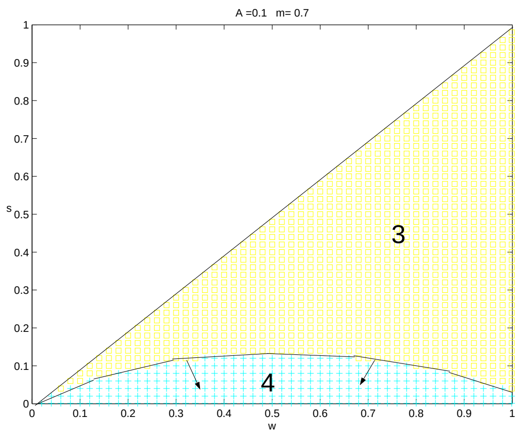


Figure 5: m = 0.6 and A = 0.1, A = 0.9, A = 0.999

Once the probability of being matched increases to $m = 0.7$, the ordering 2 completely disappears and the only type of ordering under which all the matches are accepted that persists is the ordering 3, for which the big extraction of profits from the weak agents is typical.

For high values of the parameter A the ordering 7 occurs. This ordering is also characterized by the big extraction of profits by the strong agents from the weak agents. In fact, it is so big that the weak agents prefer to stay single and wait for another match. This is why the ordering 7 occurs for the first time for a relatively high probability of being matched.



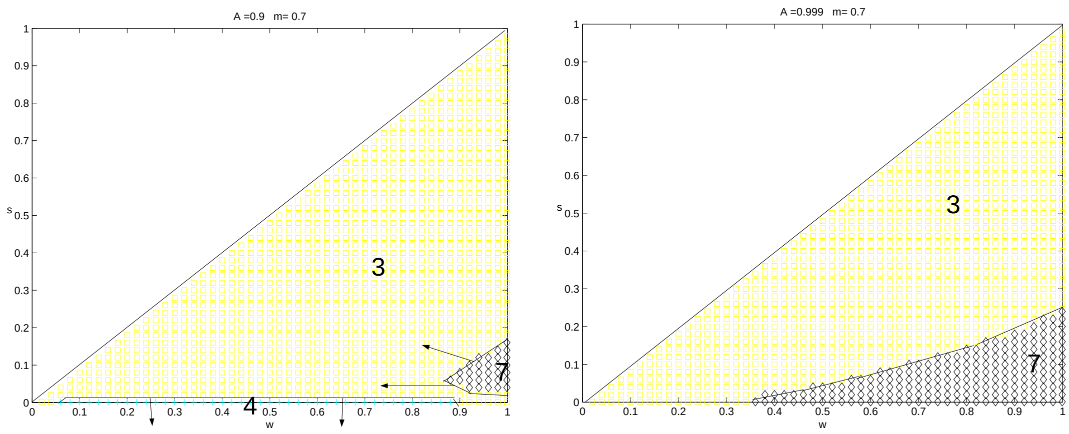


Figure 6: m = 0.7 and A = 0.1, A = 0.9, A = 0.999

As the probability of matching goes to $m = 0.8$ (Figure 7), the situation does not change drastically, the only new ordering we can observe for high factions of strong agents in the economy ($A \geq 0.6$) is the ordering 6.

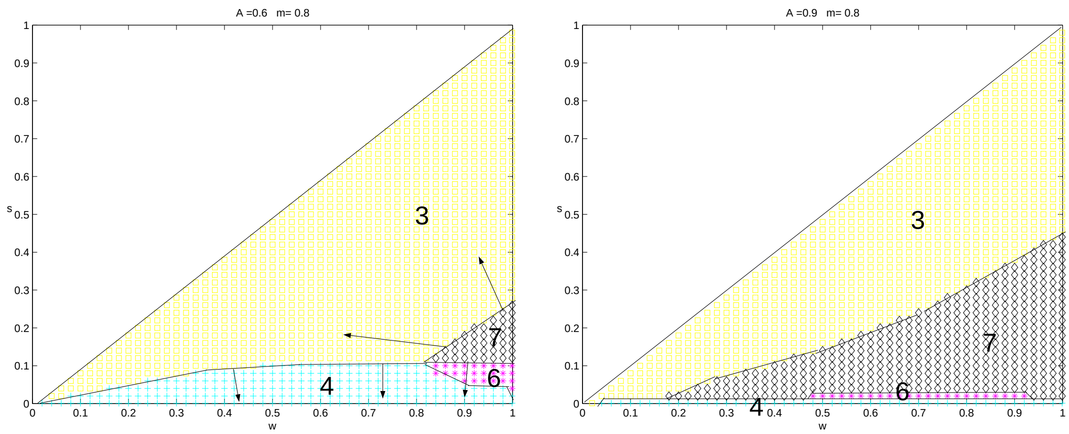


Figure 7: m = 0.8 and A = 0.6, A = 0.9

The ordering 6 is an interesting one. It corresponds to the state of the economy where the strong agents take the advantage from a big difference between the exit probabilities of strong and weak agents, and they extract from the weak agents so much that not only the weak agents reject to match with strong, but also the value of strong agents matched with another strong goes down (in relative terms) so much that the strong agents prefer to stay single rather than to be matched with their own type. This means that the weak agents reject to match with strong and at the same time the strong agents also reject to match with strong, leaving all the strong agents single, therefore having value equal to 0. This is clearly not an optimal result because if the strong agents would match with their type, the production of the whole economy would increase, and at the same time the value of both ss-type of agents and so-type would increase. The accepting of ss-type of matches would effectively lead to the same equilibrium as the ordering 7.

After another increase of the probability of being matched ($m = 0.9$) the range of parameters for which we observe accepting of all the proposed matches shrinks to such values of the probabilities of exit that are close to each other ($w - s$ is small) and therefore the ordering 3 occurs in the area close to 45° line. As demonstrated by the Figure 8, the area or the ordering 4, i.e. the area where ss- type of matching is rejected shrinks with the fraction of strong agents increasing because with A increasing the value of ss-type of match has to go up. For high values of A again the orderings 6 and 7 occur. The strong agents are using their dominant role (in terms of high fraction of strong agents as well as their low probability of exit) and try to extract profits from the weak agents.

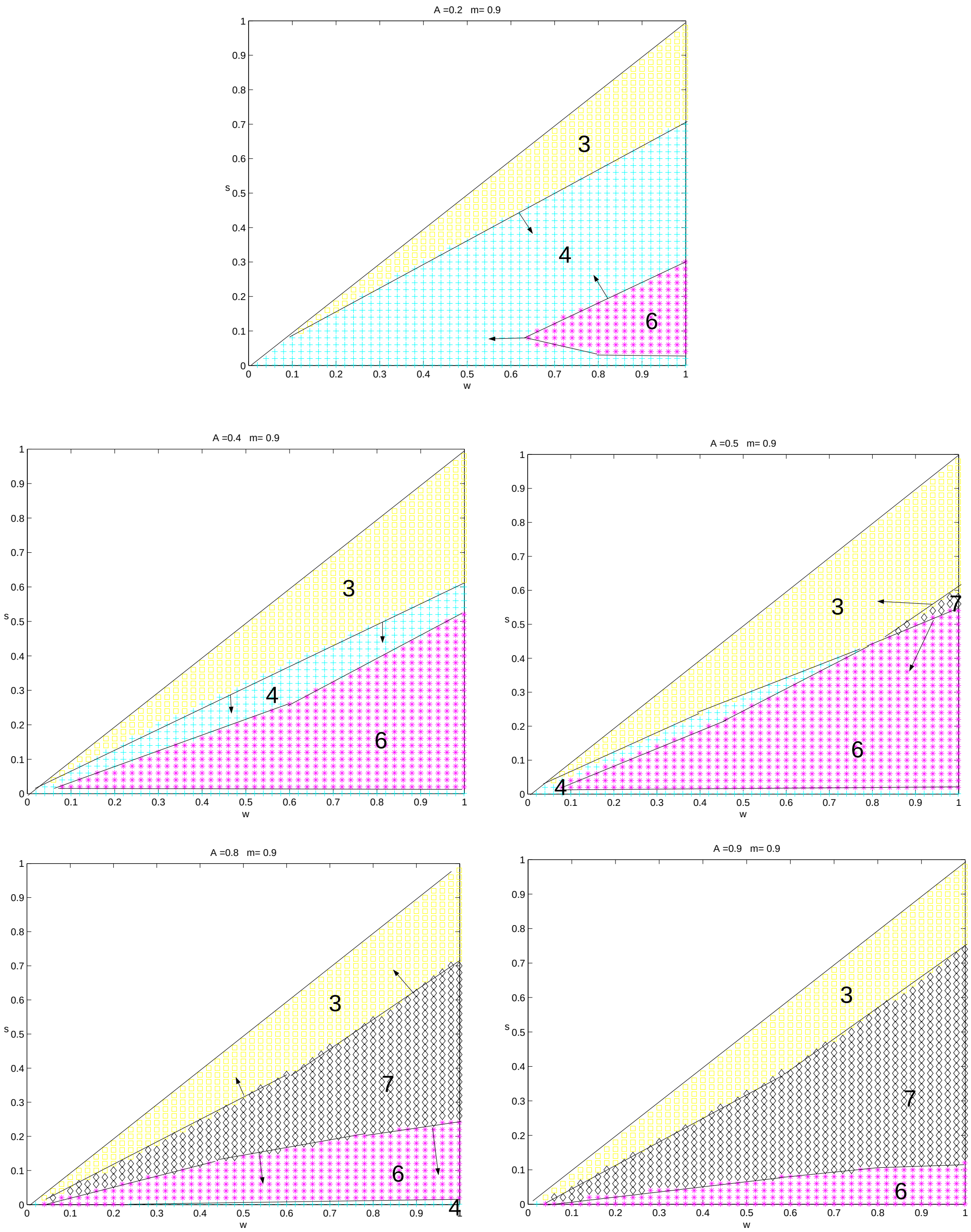


Figure 8: m = 0.9 and A = 0.2, A = 0.4, A = 0.5, A = 0.8, A = 0.9

Naturally, when there are almost no weak agents in the economy (Figure 9) only the orderings 6 and 7 occur.

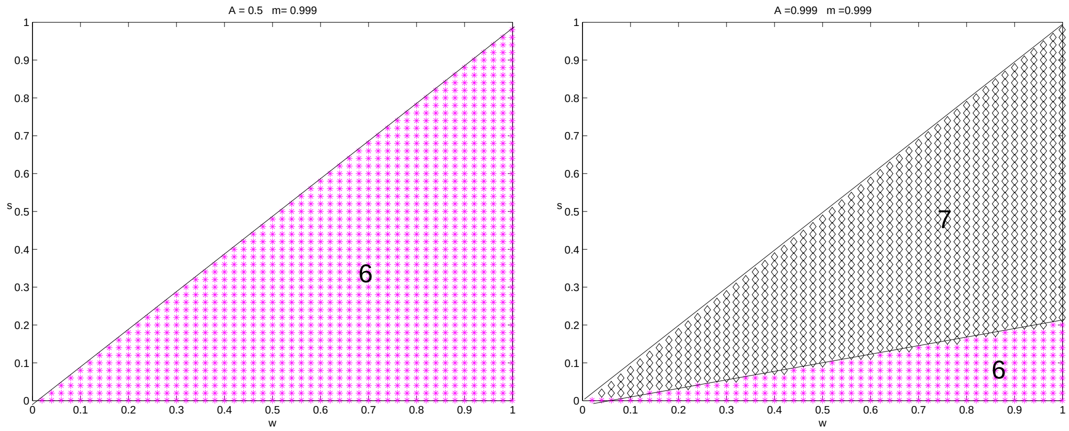


Figure 9: m = 0.999 and A = 0.5, A = 0.999

4 Conclusions

The proposed simple model of two-sided matching in an economy where agents differ in their probabilities of exit from the economy suggests the following.

When the meetings of single agents are rare any two agents that meet will indeed create a match no matter what are the values of other parameters of the model. In other words, a stable equilibrium will contain all six possible types of agents.

Once the probability of meeting a potential partner becomes higher than 50% the agents start optimally reject some of the potential partners they meet, i.e. the equilibrium with six types of agents is no longer stable for all possible combinations of parameters of the model.

One type of match is rejected in situations when meetings of single agents are relatively frequent ($m \geq 0.6$) and at the same time the share of the strong agents in the economy is not too high (so that the probability of meeting a weak agent is relatively high). Under these conditions the match of strong agents matched with their own type is unstable and therefore does not occur in equilibrium. Interestingly enough, under some values of the parameters even the equilibrium with the remaining five types of agents is not stable. This happens in situations when strong agents try to extract a lot of profits from the weak agents and these therefore reject the matches with strong. These rejections to match then lead to a stable but inefficient equilibrium in which all strong agents are single and weak agents are either single or matched with their own type. Optimally, after the rejection of mixed pairs from the side of weak agents, strong agents should reconsider their decisions and accept matches with their own type.

Another possibility is that the weak agents reject to match with strong agents that try to extract too much profit. This happens typically when meetings of singles are relatively frequent, the share of strong agents in the economy is at least 50%, and the difference between the probabilities w and s is not too big so that the matches with strong agents do not have a significantly longer expected lifetime than the matches with weak agents, which would add value to the matches with strong agents.

For certain range of parameters a combination of the previously mentioned cases can occur. If the probability of being matched reaches at least 90% and for all but extreme cases of the share of strong agents in the economy A we can observe that for some combinations of w and s the strong agents want to extract a lot of profits from weak agents, which has two effects. The first one is the already described fact that the weak agents will reject to match with strong agents and they will prefer to stay single with prospects of being matched with their own type. The second effect is that because of the extraction of profits from weak agents for the values of matched strong agents the inequality $v_{sw} \geq v_{ss}$ holds. At the same time, because the probability of being matched is high, the value of single strong agents exceeds the value of being matched with another strong. It is caused by the fact that the value of being single incorporates the prospects of being matched with the weak agents in the future, which has a high value in this particular case. That is why the strong agents will reject to matched with their own type. Again the economy will get into a stable but inefficient equilibrium with three equilibrium types: single agents of both strengths, and weak agents matched with their own type. Note that the equilibrium with five types, i.e. without the ss type, is also unstable because the weak agents will still reject to match with strong agents. On the other hand, the equilibrium with four types, i.e. with single agents and with agents matched to their own type, is always stable but by taking optimal decisions agents can not end up in this equilibrium.

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