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Existence of fair allocations in economies with production

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Abstract

This paper deals with the problem of compatibility between Pareto-optimality and equal-liberty (alias envy-freeness) in economies with production, seemingly closed after Pazner and Schmeidler's counter-examples. A simple Spence–Mirrlees-like condition (demanding that the less productive jobs are also marginally the harder in terms of well-being) is proposed, under which it is proved that allocations satisfying Pareto-optimality and equal-liberty always exist. This allows us to define a well-alive class of economies where the idea of 'maximal equal-liberty' does not lead to a paradox.

Key words: Inequality; Equal liberty; Justice

JEL classification: D63; D30

1. Introduction

Imagine a world where individuals are endowed with different innate talents and abilities which determine the market price of their labour. These individuals are also endowed with different ordinal preferences regarding labour and consumption. This paper addresses the following classical problem: how shall we define a fair allocation of labour and consumption between these individuals?

A first appealing criterion is Pareto-optimality: a (technologically-)feasible allocation of labour and consumption such that there exists another allocation which every individual prefers to the former but which is not

acceptable. This criterion is obviously not sufficient: many allocations are Pareto-optimal, and some of them violate any reasonable view of equity; for example, the *laissez-faire* market equilibrium, where everyone chooses one's labour supply given the price one can obtain on the labour market, which may lead to misery for those who have been disadvantaged by nature and the current technology, and to luxury for some others. Recall indeed that by definition the individual differences in productive abilities that we are concerned with do not come from rational choices made by identical agents facing initially the same opportunities; on the contrary, these individual differences must be interpreted as the outcome of some exogenous and innate heterogeneity. Obviously, a simple (but unconvincing) way to solve the problem would be to deny the existence of such heterogeneities (for example, this is what the concept of 'wealth-fairness' does implicitly; see Section 2).

A particularly natural concept of equity in such a world is that of 'equal-liberty': an allocation will be said to be 'fair' if, on the one hand, it satisfies Pareto-optimality, and if, on the other hand, it is the outcome of a choice process where every agent had the liberty to choose his pair (labour–consumption) in the same opportunity set, regardless of the exogenous inequalities characterizing these individuals (i.e. 'labour' is measured by the same universal unit for everybody, such as labour time).¹ The ethical justification of this 'equal-liberty' concept is that every human being should have access to the same life opportunities, regardless of innate productivity differentials.

Thus this equity concept attempts to capture the idea of 'maximal equal-liberty': every individual must have access to the same opportunity set, and the resulting state of the world must be such that no individual can be made better-off without hurting another individual. It is straightforward that the *laissez-faire* market equilibrium is not fair according to this definition (individual opportunity sets depend on the extent to which one was favored by nature). The question we address in this paper is the existence of such allocations.

The paradox faced by the study of this concept is indeed that there exists worlds where equal-liberty and Pareto-optimality exclude each other: Pazner and Schmeidler (1974) displayed economies where equal-liberty implies the existence of some Pareto-improvement. However the generality and the meaning of these examples do not seem to have been studied in more detail (see, however, Tillman, 1989, and Section 2), and their main

¹ It may be more consistent to use the word 'fair' only to refer to the 'equal-liberty' property, since Pareto-optimality is not strictly speaking a fairness-type property. For simplicity, however, and following a common practice, we use the word 'fair' when both equal-liberty and Pareto-optimality are verified

effect has been to reduce substantially the academic interest for this demanding concept of equity in production economies (see Section 2 for a brief review of this literature).

Our objective in this paper is to propose a natural condition linking individual differences in productivities and individual differences in labour–consumption preferences under which equal-liberty and Pareto-optimality do not exclude each other any more, i.e. under which there always exists fair allocations of labour and consumption. This condition requires that agents endowed with lower productive abilities have lower marginal rates of substitution between leisure and consumption, i.e. that they judge their labour more costly in terms of well-being. The existence result is based on the property that in such economies there cannot exist Pareto-efficient ‘envy cycles’, which implies directly the existence of fair allocations via the Knaster–Kuratowski–Mazukiewicz lemma (also used by Varian, 1974, for the case of pure exchange economies).

We do not claim that this Spence–Mirrlees-like condition is necessarily verified in social practice, but it allows us, however, to define in a simple and intuitive manner a class of economies in which the idea of ‘maximal equal-liberty’ does not lead to a paradox. Moreover, this class of economies is undoubtedly well-alive: it is not rare to observe that jobs associated with the lowest wage rates (say, unskilled factory worker) are also those which are considered as more costly in terms of well-being by individual preferences (in the sense that working long hours is so hard that individuals have marginally a higher propensity to trade off more leisure against less consumption).

The rest of this paper is organized as follows: Section 2 makes more precise the link between our approach and contribution and other theories and concepts of equity and justice. Section 3 introduces the formal model and gives definitions. Section 4 states and proves the general existence proposition. Section 5 gives concluding comments.

2. Relation to the literature on fairness

The objective of this section is obviously not to survey the voluminous literature on distributive justice and fairness, but rather to provide some basic background for the understanding of our contribution.

Distributive justice as ‘maximal equal-liberty’ has stimulated a large body of research (see Thomson and Varian, 1985, for a survey; see also Kolm, 1993a,b, for a recent synthesis).

Note first that the ‘equal-liberty’ concept is sometime referred to as the ‘non-envy’ criterion. It is straightforward indeed that the equal-liberty property (the allocation is the outcome of a choice process where each

individual faces the same opportunity set) is fully equivalent to the non-envy property (each agent prefers his own labour–consumption) bundle to those of other agents) as long as individual agents are rational (facing an opportunity set they always choose the bundle they prefer the most). However it is essential to distinguish between the equal-liberty property, which constitutes the ethical foundation of this justice concept, and the non-envy property, which is at best a formal coincidence and at worst a complete non-sense (since feelings of envy may be considered as interesting per se only if they take the form of an externality, i.e. if individual well-beings depend explicitly on other agents' bundles; see Kolm, 1991).

In the context of pure exchange economies, this concept of justice does not meet any existence problem: the market equilibrium resulting from an equal division of initial physical resources verifies the equal-liberty property and Pareto-optimality (under standard convexity assumptions). In fact, various assumptions even imply that the unique fair allocations of pure exchange economies are these 'equal-endowment market equilibria' (see Varian, 1976).

Unfortunately this does not hold any more if one considers production economies where individual agents are endowed with various productive abilities. The logic of Pazner and Schmeidler (1974)'s examples of non-compatibility between equal-liberty and Pareto-optimality is clearly summarized by Amartya Sen:

Pareto-optimal allocations may require that the more productive should work harder and be paid more. And the leisure-loving more productive may, in this situation, envy the unhurried less productive, while the income-loving less productive may envy the opulent more productive (Sen, 1986, p. 1108).

As was mentioned above, the compatibility problem revealed by Pazner and Schmeidler (1974) led many theorists to develop new concepts of distributive justice (see, for example, Varian, 1974; Daniel, 1975; Varian, 1975; Pazner, 1977; Pazner and Schmeidler, 1978a,b; Feldman and Weiman, 1979; Thomson, 1982; see also the survey by Thomson and Varian, 1985). For example, the concept of 'wealth-fairness' proposed by Varian (1974) defines equal-liberty in terms of equal access to a set of (individual output–consumption) bundles: as Varian writes, 'I may 'envy' a doctor who only works one day a week doing brain surgery and yet has a substantial consumption; but unless I am willing to put in enough labour time to match his production of services – for example, 6 years of medical school required – my complaint against him cannot count as legitimate in the sense of equity' (p. 73). Thus such a theory of fairness gives up any attempt to correct inequalities in abilities (in particular the competitive equilibrium resulting from an equal division of physical resources satisfies 'wealth-fairness'). However, recall that by definition we are interested in the

'residual' differences in productivities which are entirely due to exogenous differences in natural talents (as opposed to the human capital differentials referred to by Varian), and it has been argued for some time that such differences in innate talents are irrelevant from the viewpoint of social ethics (see, for example, Rawls, 1971). Thus the wealth-fairness concept 'solves' the compatibility problem at a very high cost.

Other substitutes for the initial 'equal-liberty' concept (such as that of 'income-fairness' proposed by Varian 1974, and Pazner and Schmeidler, 1978b, or that of 'egalitarian-equivalence' proposed by Pazner and Schmeidler, 1978a) do not necessarily suffer from this criticism, but in general they have other drawbacks (for a review and subsequent references, see Thomson and Varian, 1985, pp. 117–120). Although we do not suggest that none of these concepts succeeded in providing an interesting alternative to the initial 'equal-liberty' concept, we believe that the latter is sufficiently attractive to justify an analysis of sufficient conditions under which Pazner and Schmeidler (1974)'s counter-examples break down. Surprisingly enough, and as far as we know, this has not been done: the only extension of Pazner and Schmeidler's work that we are aware of has been done by Tillman (1989), where it is essentially proved that if we allow for a sufficiently large domain of individual preferences and productivity differentials then the non-existence result shows up. By definition the sufficient condition for existence that we are proposing violates the assumption of Tillman (1989)'s negative results (see Section 3), but we believe that this condition is nevertheless interesting since it defines a simple and well-behaved class of economies where 'maximal equal-liberty' makes sense.

One attractive feature of the fairness literature referred to so far is that no interpersonal comparison of well-being is ever invoked (i.e. these are purely ordinal theories of distributive justice). One may want, however, to apply non-ordinal theories of justice to our framework in order to compare the resulting allocations with our 'fair' allocations (i.e. satisfying equal-liberty and Pareto-optimality).

For example, according to John Rawls' theory (see Rawls, 1971), social and economic inequalities should be arranged so as to satisfy the 'Difference Principle' (i.e. so as to benefit to the most disadvantaged members of the society), which requires the objective knowledge of the identity of the most disadvantaged members of the society. However, in general there is no obvious way to obtain such a knowledge in a world where individuals differ not only by their productive talents but also by their ordinal preferences regarding labour and consumption; thus unless we make further interpersonal-comparability assumptions the Rawlsian allocation of our basic distributive justice problem is not well-defined. We believe that one considerable strength of the equal-liberty concept is precisely to allow for a consistent and ordinal definition of fairness in situations where the Differ-

ence Principle raises non-trivial comparability problems (i.e. outside the case where individuals share common ordinal preferences regarding labour and consumption)². Note that although Rawls' second principle of fairness (i.e. the Difference Principle) is explicitly designed to determine fairness in economic inequalities, one may wonder whether the latter is really necessary and relevant: a broad interpretation of the domain of social interactions over which Rawls' first principle of fairness ('Each person is to have an equal right to the most extensive liberty compatible with a similar liberty for others', p. 60) ought to be used leads us very close to the concept of 'maximal equal-liberty' that we study in this paper.

One may also want to compare our 'fair' allocations to the allocations prescribed by utilitarianism. As usual with utilitarianism, the social optimum depends on the particular cardinal representation that one takes for each individual ordinal preference, and any comparison with 'fair' allocations will depend on the choice one makes at this stage (note that since ordinal preferences are different it is not only a problem of observing individual risk aversions but also of rescaling individual utilities). Apart from these technical difficulties there are fundamental reasons why utilitarianism cannot be considered as an attractive theory of distributive justice (we refer to Kolm, 1992, for a recent criticism of the ethical foundations of utilitarianism), as opposed to the 'equal-liberty' theory.

3. Model and definitions

We consider an economy inhabited by a set of agents $I = (1, \dots, n)$. There exists a finite set of labour types $J = (1, \dots, p)$ and a unique consumption good C . We assume the technology to be linear and separable, i.e. each labour type $j \in J$ has a constant productivity $\theta(j) > 0$ in terms of the consumption good C (more generally, $\theta(j)$ is the competitive wage rate of type j labour market, and thus reflects the general structure of technology and demand, which we summarize by this reduced form for simplicity). Without any loss in generality, we assume that

$$\theta(1) < \theta(2) < \dots < \theta(p).$$

Nature distributed talents in such a manner that each agent $i \in I$ can

² If all individual agents have the same ordinal preferences regarding labour and consumption then one can compare individual levels of well-being. This allows us to define the Rawlsian optimum (which is simply the unique Pareto-efficient allocation where every agent is on the same indifference curve), which coincides with the unique Pareto-optimal allocation satisfying the equal-liberty property.

supply only one type of labour $j(i) \in J$ (alternatively, one could assume that $j(i)$ is the most productive labour type that i can supply; this makes no difference for the existence problem). Each agent $i \in I$ has well-behaved preferences over (consumption good C ; labour type $j(i)$) bundles, which can be represented by a strictly quasi-concave, strictly monotonic and continuous utility function $U_i(C, l)$ defined over $R^+ X[0; L]$, where L is the maximum amount of labour (measured in time units) the agents can supply (i.e. L is the same for every agent), l is the quantity of labour type $j(i)$ supplied by agent i (measured in time units), and C is the amount of the consumption good consumed by agent i . Without any loss in generality, we assume that

$$j(1) \leq j(2) \leq \dots \leq j(n).$$

An allocation of labour and consumption $((C(i), l(i)), i \in I)$ is (technologically-) feasible if and only if

- (a) $\forall i \in I, (C(i), l(i)) \in R^+ X[0; L],$
- (b) $\sum_{i \in I} C(i) \leq \sum_{i \in I} \theta(j(i))l(i).$

A feasible allocation is Pareto-optimal if and only if there exists no other feasible allocation where at least one agent is better off and the others are not worse off.

An opportunity set for an agent $i \in I$ is a set A_i of pairs $(C, l) \in R^+ X[0; L]$ within which i has the liberty to choose the pair he prefers (agents are assumed to be rational welfare maximizers).

An allocation $((C(i), l(i)), i \in I)$ verifies the equal-liberty property if it is the outcome of a choice process where every agent i had access to the same opportunity set $A_i = A, \forall i \in I$.

It is straightforward that an allocation $((C(i), l(i)), i \in I)$ verifies the equal-liberty property if and only if $((C(i), l(i)), i \in I)$ is envy-free, in the (loose; see Section 2) sense that no agent prefers the pair (C, l) of another agent to his own pair:

$$\forall i, i' \in I, U_i(C(i), l(i)) \geq U_i(C(i'), l(i')).$$

A feasible allocation will be said to be fair if it satisfies both Pareto-optimality and the equal-liberty property (see Footnote 1). In general, one can find some combinations $((\theta(j))_{j \in J}, (j(i), U_i(C, l))_{i \in I})$ of individual productivities and preferences such that the set of fair allocations is empty (see Pazner and Schmeidler, 1974, and Tillman, 1989).

As was announced above, we will make the following additional assumption, which is a sort of 'generalized Spence–Mirrlees condition':

$$\text{GSM condition: } \forall i, i' \in I, (j(i) < j(i')) \rightarrow (\forall (C, l) \in R^+ X[0; L] \\ -\partial_l U_i(C, l) / \partial_c U_i(C, l) \geq -\partial_l U_{i'}(C, l) / \partial_c U_{i'}(C, l)).$$

This assumption means that, at any bundle, a worker with a lower wage will accept to work a smaller amount of time for consuming one more unit than a worker with a higher wage (the wages, of course, do not enter directly in this marginal deal proposed to the agents). This expresses the idea that we are concerned with economic environments where the agents with a low wage do a harder work than the agents with a high wage, i.e. that one hour of work for, say, an unskilled factory worker is more costly than one hour of work for, say, a doctor as far as their well-being is concerned.³ Note that the inequality involved in the GSM condition is not strict, so that a special case of our analysis is the much-studied one where all agents have the same utility function.

Note also that in the example of Sen's quotation above this condition is not satisfied, as the less productive agents are assumed to be also the more 'income-loving' (this expression is an imperfect summary for the exact marginal property). The examples of Pazner and Schmeidler (1974) rely on this violation of GSM. In the same manner, Tillman (1989)'s non-existence results say that if one considers a family of utility functions indexed by their (constant) ordering of MRSs, then the set of fair allocations is empty if the underlying domain of economic fundamentals is the cartesian product of some interval of linear productivities by this family of utility functions (which implies that GSM is violated). In fact, it turns out (unsurprisingly) that every example displaying inconsistency between Pareto-optimality and the envy-free criterion must violate the GSM condition, and that this condition is sufficient for ruling out this incompatibility, as we are now going to state and prove.

4. Existence of fair allocations

As was announced in Section 1, we will first prove that in an economy satisfying the GSM condition there cannot exist any Pareto-optimal 'envy-cycle', and then apply the KKM lemma. Note that there exists a direct proof, which may even be slightly shorter; however the two-step proof we give has two advantages. First it is the exact equivalent of the proof commonly used for the case of pure exchange economies (see Varian, 1974), and thus it allows us to clarify the link between production and exchange economies regarding the existence problem: in pure exchange economies the non-existence of Pareto-optimal envy cycles is a straightforward property

³ Note, however, that if this ordering of MRS seems obvious on an intra-personnel basis (for one more consumption unit, a doctor would certainly accept working for a shorter period of time as an unskilled factory worker than as a doctor), it is less obvious on an inter-personnel basis (as demanded by the GSM condition), although realistic.

(and therefore so is the existence of fair allocations), whereas Pareto-optimal envy cycles may exist in production economies. Secondly the non-existence of envy cycles has been considered as interesting per se by a number of authors (and is not implied a priori by the existence of fair allocations): Varian (1974) suggests an interpretation of Rawls' worst-off agent as the one whose bundle is envied by no one (as opposed to the best-off agent who envies the allocation of nobody), and the existence of such agents is guaranteed at any Pareto-efficient allocation by the non-existence of Pareto-efficient envy cycles.

An allocation $((C(i), l(i)), i \in I)$ is an envy cycle if and only if there exists $r, i(1), i(2), \dots, i(r) \in I$ such that

$$\forall s \in (1, \dots, r), \quad U_{i(s)}(C(i(s)), l(i(s))) < U_{i(s)}(C(i(s+1)), l(i(s+1)))$$

(with $i(r+1) = i(1)$).

Proposition 1. In an economy satisfying the GSM condition, there exists no Pareto-optimal envy cycle.

Proof. We will prove Proposition 1 by induction on the length r of the cycle. Let us define

- (a) $\forall i \in I, T_i = [-\theta(j(i))L; +\infty[$,
- (b) $\forall i \in I, (C_i, l_i): T_i \rightarrow R^+ X[0; L]$
 $T \in T_i \rightarrow \operatorname{argmax} U_i(C, l)$ under $C \leq \theta(j(i))l + T$,
- (c) $T = ((T(i), i \in I)$ such that (1) $\forall i \in I, T(i) \in T_i$,
- (2) $\sum_{i \in I} T(i) = 0$).

These mappings are well-defined thanks to the assumptions on preferences. The second welfare theorem tells us that the set of Pareto-optimal allocations (as well as the set of Pareto-optimal utility profiles) is homeomorphic to T , via the mappings $(C_i, l_i)(T)$.

Let us define, for any $r \in I$, the proposition $P(r)$ by

P(r). In any economy satisfying the GSM condition, there exists no Pareto-optimal cycle of length r .

$P(1)$ is trivial. Let us prove $P(2)$ directly:

Lemma 1. $\forall i, i' \in I, T \in T_i, T' \in T_{i'}$, at least one of the following statements holds true:

- (a) $U_i(C_i(T), l_i(T)) \geq U_{i'}(C_{i'}(T'), l_{i'}(T'))$,
- (b) $U_{i'}(C_{i'}(T'), l_{i'}(T')) \geq U_i(C_i(T), l_i(T))$.

Proof of Lemma 1. First note that if $j(i) = j(i')$, then the lemma is trivial: if,

say, $T > T'$, then i' 's opportunities set (i.e. budget set) is included in i 's budget set, and therefore i cannot envy i' .

Now, let us assume, without any loss in generality, that $j(i) < j(i')$, and that (a) and (b) are both false. Assume first that $l_i(T) > l_{i'}(T')$. As $(C_i, l_i)(T)$ is above i' 's indifference curve I' passing through $(C_{i'}, l_{i'})(T')$, and $(C_{i'}, l_{i'})(T')$ is above i 's indifference curve I passing through $(C_i, l_i)(T)$, then I must cross I' at a point at the north-east of $(C_{i'}, l_{i'})(T')$. It then follows from strict quasi-concavity of preferences that

$$\begin{aligned} \theta(j(i)) &= -\partial_l U_i(C_i(T), l_i(T)) / \partial_C U_i(C_i(T), l_i(T)) \\ &> -\partial_l U_{i'}(C_{i'}(T'), l_{i'}(T')) / \partial_C U_{i'}(C_{i'}(T'), l_{i'}(T')) = \theta(j(i')), \end{aligned}$$

which is impossible, as $j(i) < j(i')$.

Suppose now that $l_i(T) < l_{i'}(T')$. For the same reasons, I must cross I' at the south-west of $(C_{i'}, l_{i'})(T')$ with a slope smaller than the slope of I' at this point, which is contradictory with the GSM condition (see Fig. 1). \square

Now we need another simple lemma to complete the proof of Proposition 1:

Lemma 2. $\forall i, i', i'' \in I, T \in T_i, T' \in T_{i'}, T'' \in T_{i''}$, such that

- (a) $\theta(j(i)) \geq \theta(j(i'))$, $\theta(j(i)) \geq \theta(j(i''))$,
 - (b) $U_i(C_i(T), l_i(T)) < U_{i'}(C_{i'}(T'), l_{i'}(T'))$,
 - (c) $U_{i''}(C_{i''}(T''), l_{i''}(T'')) < U_{i'}(C_{i'}(T'), l_{i'}(T'))$,
- then $U_{i''}(C_{i''}(T''), l_{i''}(T'')) < U_{i'}(C_{i'}(T'), l_{i'}(T'))$.

Proof of Lemma 2. First note that the case where equalities hold true in (a) is just as trivial as in Lemma 1. Therefore let us assume that strict inequalities hold true. Consider I, I', I'' the indifference curves as defined in the proof of Lemma 1, and assume that $(C_{i'}, l_{i'})(T')$ is below I'' . Now two things can happen: either I'' crosses I at the north-east of $(C_i, l_i)(T)$, or it does so at the south-west. In the second case, i'' does not envy i because of the GSM property; in the first case, quasi-concavity of preferences implies that the slope of I' at $(C_{i'}, l_{i'})(T')$ is higher than the slope of I at $(C_i, l_i)(T)$, which contradicts $j(i) > j(i')$ (see Fig. 2). \square

We can now prove $P(r)$ by induction. Consider $r > 2$, assume that $P(r)$ is true, and that there exists a Pareto-optimal envy cycle of length $r + 1$. Let i be one of the agents of the cycle with the highest productivity, and let i' be the agent envied by i , and i'' the agent who envies i . Then Lemma 2 implies that i'' envies i' , and therefore one can cancel i from the cycle and still have an envy cycle of length r . But this contradicts $P(r)$, and thus the proof is completed. \square

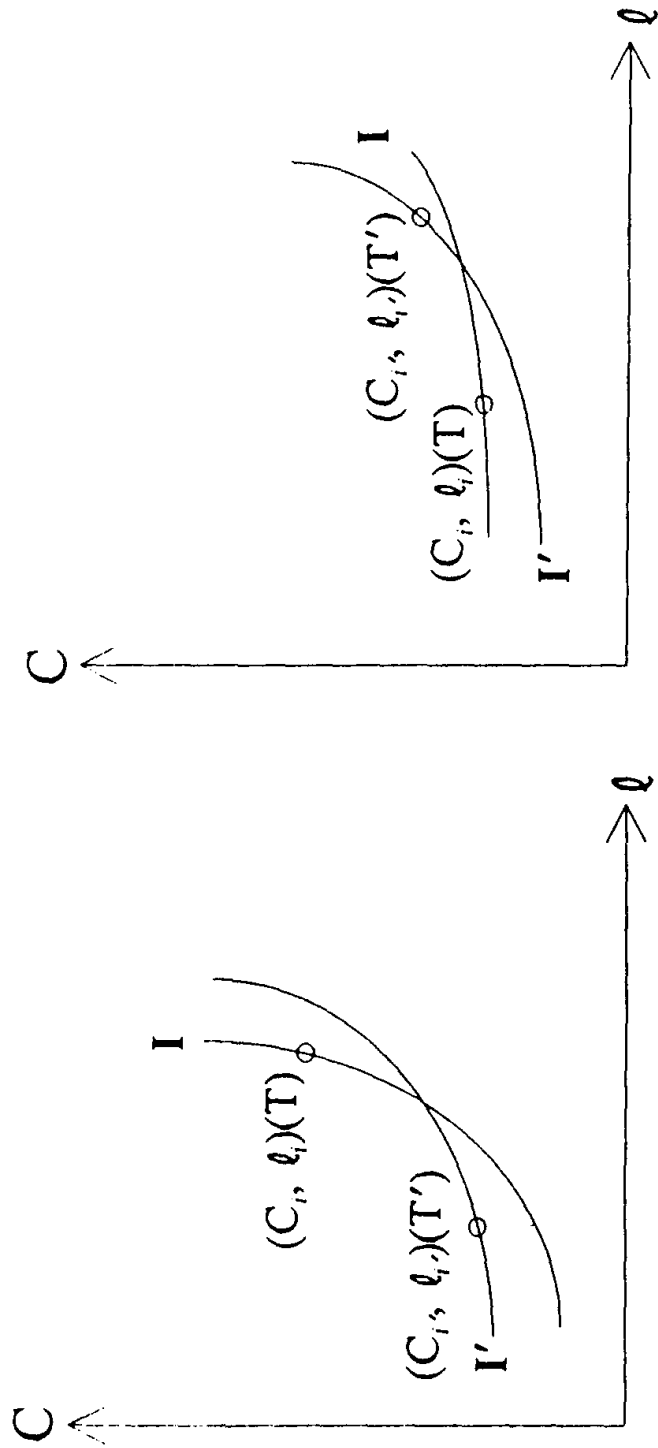


Fig. 1.

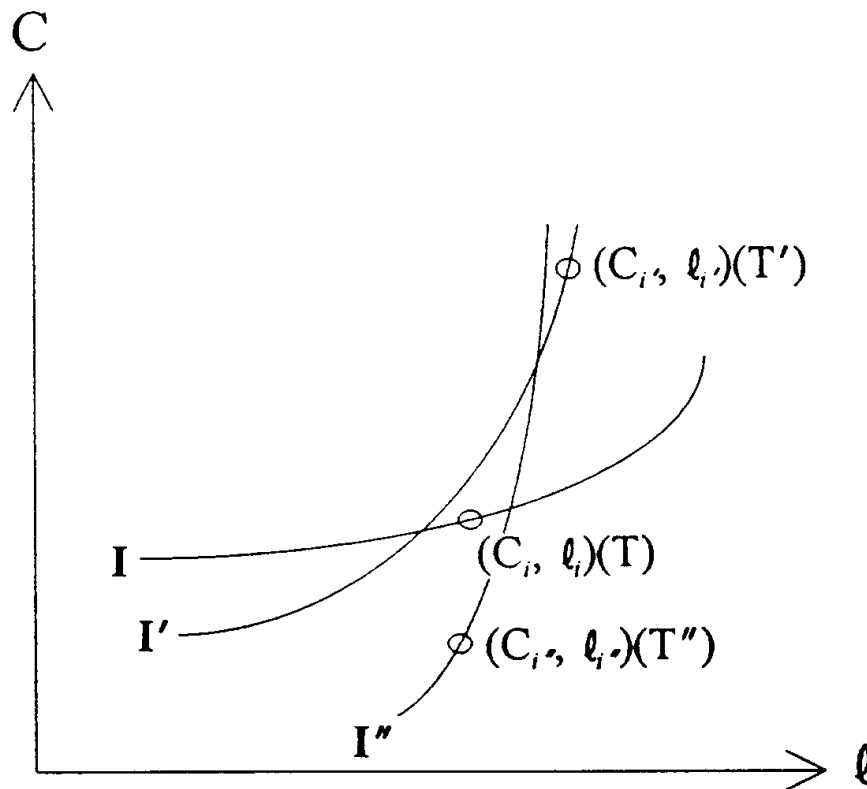


Fig. 2.

It is worth noting that, as one can check, the GSM condition is not fully necessary for the proofs of Lemmas 1 and 2. However, we did not find any reasonably simple weaker condition implying the impossibility of efficient envy cycles in economies with production.

As it is precisely stated above, the existence of fair allocations is a direct consequence of the Knaster–Kuratowski–Mazurkiewicz lemma in an economy satisfying Proposition 1 (as it was noted by Varian (1974) for the case of pure exchange economies, where the non-existence of Pareto-optimal envy cycles is trivial). The Knaster–Kuratowski–Mazurkiewicz lemma is simply a generalization of the following intuition: consider the special case where $n = 2$, $j(1) < j(2)$; at the competitive equilibrium without lump-sum transfers, agent 2 does not envy agent 1, but agent 1 is very likely to envy agent 2 (if he does not, then this allocation is fair); but for a sufficiently high positive transfer from agent 2 to agent 1, agent 2 will eventually envy agent 1 while agent 1 will not envy agent 2 any more at the associated competitive equilibrium allocation; using a continuity argument, one can conclude that between these two efficient allocations there must exist some efficient allocations where both are non-envious at the same

time, as they cannot both be envious at the same time according to Proposition 1.

Proposition 2. In any economy satisfying the GSM condition, the set of fair allocations is non-empty.

Proof. As was mentioned above the set U of Pareto-optimal utility profiles is homeomorphic to T ; T itself is homeomorphic to the $n - 1$ -dimensional simplex S^{n-1} , the natural mapping being

$$T \rightarrow S^{n-1},$$

$$(T(i), i \in I) \rightarrow \left(x_i = (T(i) + \theta(j(i))L) / \left(\sum_{i' \in I} \theta(j(i'))L \right), i \in I \right)$$

Therefore U and S^{n-1} are homeomorphic, via a mapping F . For $i \in I$, let us define $M(i)$ by

$$M(i) = \{ u = (u_{i'}, i' \in I) \in U \text{ s.t. } \forall i' \in I, u_{i'} \geq u_i \}.$$

Since Proposition 1 implies in particular that at any efficient allocation there is at least one agent who no one envies (if not, one can form an efficient envy cycle), U is contained in the union of the $M(i)$. Via F , S^{n-1} is contained in the union of the $F(M(i))$. Moreover, for any $i \in I$, $F(M(i))$ contains the i th face of the simplex, because the bundle obtained by agent i with a negative transfer of $\theta(j(i))L$ cannot be envied by anyone (as this bundle is the worst one $(0, L)$). Finally, the $M(i)$ are closed subsets of U by continuity, and therefore the $F(M(i))$ are closed subsets of S^{n-1} , because a continuous function maps compact sets onto compact sets.

We can now apply the Knaster–Kuratowski–Mazurkiewicz lemma:

Lemma (Knaster, Kuratowski, Mazurkiewicz). Let $M(1), \dots, M(n)$ be a family of closed subsets of the $n - 1$ -dimensional simplex S^{n-1} with the property that the i th face of S^{n-1} is contained in $M(i)$, and that S^{n-1} is contained in the union of the $M(i)$. Then the intersection of the $M(i)$ is non-empty (see Scarf, 1973, p. 68).

By applying F^{-1} to the non-empty intersection of the $F(M(i))$ it follows that the set of Pareto-optimal envy-free allocations is non-empty. \square

5. Concluding comments

The economic structure in which we studied the problem of distributive justice in this paper is highly primitive for many reasons: in particular it does

not integrate any inter-generational or dynamic dimension, nor complementarities in production, which constitute notable loci of tension for the idea of justice as 'equal-liberty'. In clear, it is uncertain whether the difficulties met by the idea of justice in the framework we considered are the most relevant which show up in practice.

We believe, however, that this paper illustrates one possible way of going beyond the usual impossibility results of social choice theory without leaving the most compelling concepts of distributive justice: instead of attempting to find possibility theorems holding for all logically-possible individual preferences, or using some mathematical concept of genericity to restrict the latter, it may make sense to consider domain restrictions based upon some clear, socially intuitive and easy-to-check conditions (such as the sufficient condition we proposed).

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