COMMUNICATIONS

The Golden Rule of Accumulation: A Fable for Growthmen

Once upon a time the Kingdom of Solovia was gripped by a great debate. "This is a growing economy but it can grow faster," many argued. "Sustainable growth is best," came the reply, "and that can come only from natural forces."

A few called the debate growthmanship. But most thought it would be healthy if it led to a better understanding of Solovian growth. So the King appointed a task force to learn the facts of Solovian economic life.

The committee reported that the labor force and population in Solovia grew exponentially at the rate $\gamma$. The number of working Solovians, $N_t$, at time $t$ was therefore given by

$$N_t = N_0 e^{\gamma t}, \quad \gamma > 0.$$

The report expressed confidence that Solovia's supply of natural resources would remain adequate. It portrayed a competitive economy making full and efficient use of its only scarce factors, labor and capital, in the production of a single, all-satisfying commodity. Returns to scale were observed to be constant, and capital and labor were found to be so substitutable that fears of technological unemployment were dismissed.

The committee described the steady progress in Solovia's ways of production. It estimated that the efficiency of Solovian capital was increasing at the rate $\lambda$ and that Solovian labor was improving at the rate $\mu$. A continuation of these rates of technical advance was anticipated. Therefore production $P_t$, at
ciency, equity and effectiveness. Mathematicians, leading the quest for a growth strategy, grappled with extremals, functionals and Hamiltonians. Yet nothing practicable emerged.

Then a policy-maker was heard to say, "Forget grand optimality. Solovians are a simple people. We need a simple policy. Let us require that the fraction of output accumulated be fixed for all time, that is:

\[
(3) \quad \frac{dK_t}{dt} = sP_t, \text{ for all } t, 0 \leq s \leq 1.
\]

If we make investment a constant proportion of output, our search for the idea investment policy reduces to finding the best value of \( s \), the fixed investment ratio."

"It's fair," Solovians all said. The King agreed. So he established a prize for the discovery of the optimum investment ratio. The prize was to be a year abroad to learn how advanced countries had solved the growth problem.

Soon a brilliant peasant, Oiko Nomos, claimed the prize. Solovians laid down their tools, picked up pencils and pads, and conversed on their capital.
nite beginning and which has always enjoyed golden-age growth at the natural rate. It has traveled unswervingly up a single exponential path, a path stretching back indefinitely into the past. Along this path the output rate at any specified time (though not the rate of growth) depends, in general, upon the value of the equilibrium capital-output ratio. But this ratio depends
To find the value of $s$ which maximizes $C_t$, we take the derivative with respect to $s$ and equate it to zero. This yields:

$$-f(s)e^{st} + (1-s)f'(s)e^{st} = 0. \tag{8}$$

"It is apparent that upon dividing (8) by $e^{st}$ all terms involving $t$ vanish. The solution of equation (8) is therefore independent of the 'generation' whose consumption we choose to maximize. The $s$ which is optimal for one generation in a natural boundless golden age is optimal for all. This proves the lemma."

Cries of "What a lemma!" resounded in the capital and Oiko was heartened by the reception. Anticipation ran high when he moved to speak again.

"And now I wish to announce a new and fundamental theorem. Theorem: Among the optimal golden-age path, under conditions of natural growth, the rate of investment is equal to the competitive rate of profits."

"Choosing the best value of $s$ is simple enough in principle. A high value of $s$ will be associated with a high golden-age output path. But too high a value of $s$ will leave too little output available for consumption. Characterizing the exact optimum is a matter of calculus."

"Rewriting (8) in the form:

$$\frac{s}{1-s} = \frac{f'(s)s}{f(s)} \tag{8'}$$

we find that the optimal ratio of investment to consumption equals what we
Upon rearranging terms and using the capital-output relation (5) we find that

\[
\frac{f'(s)s}{f(s)} = \frac{a}{1-a}, \quad \text{where } a = \frac{F_K(K_0, N_0)K_0}{P_0}.
\]

"Looking at (8') and (11) we see easily that

\[
s = a
\]

In competitive Solovia the variable \(a\) measures capital's relative share in total output at time zero. Now we have observed that the elasticity of golden-age output with respect to the investment ratio is everywhere equal on any particular golden-age path; it follows by (11) that \(a\), the profit-income ratio, must also be constant along any particular golden-age path. Therefore, by (12), on the optimum natural growth path the investment ratio and the profit ratio are constant and equal. This proves the theorem.

"We may call relation (12) the golden rule of accumulation, and with good reason. In a golden age governed by the golden rule, each generation invests on behalf of future generations that share of income which, subject to
where $\alpha$, a fixed parameter, was the elasticity of output with respect to the capital stock. They preferred to write it in the form:

\[(2'') \quad P_t = A e^{\rho t} K_t^{1-\alpha} N_t, \quad \text{where} \quad \rho = \alpha \lambda + (1-\alpha)\mu.\]

Solovians knew then they could have any capital-output ratio they desired, with full employment. The existence of a full-employment, golden-age equilibrium for every investment ratio was assured. Differentiating logarithmically, they quickly calculated from (1) and (2'') that in a golden age, capital and output would grow exponentially at the rate $\frac{\rho + (1-\alpha)\gamma}{1-\alpha}$, independently of the investment ratio. Thus did Solovia discover her natural rate of growth. What a triumph for Oiko. His assumptions were completely vindicated.

Joyously, the Solovians hurried to compute the golden-rule path. It did not take them long to realize that $\alpha$ was capital's share. On the golden-rule path, $s$ would equal $\alpha$. Next, using (5), they divided $\alpha$ by their natural growth rate to obtain the capital-output ratio on the golden-rule path. To their great relief, the resulting ratio exceeded their actual capital-output ratio by only a small factor. No wonder for they had invested most of their profits and consumed most of their wages anyway.

With Oiko's inspiring words still ringing in their ears, the Solovian people pressed the King for a program to attain the golden-rule path. So the King proclaimed golden-rule growth a national purpose and instituted special levies. Once the golden-rule path was reached, investment was continuously equated to profits and Solovians enjoyed subject to (3) maximum social...